Combinatorics of symplectic invariant tensors

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Dedicated to Mia & George

Cyclic Sieving Phenomenon

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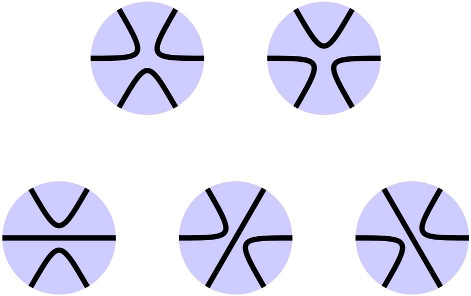
- The polynomial P determines (X, c) up to isomorphism.
- The constant term is the number of orbits.
- Evaluation at $q = 1$ gives $|X|$.

Example: Catalan



$$P(q) = 2 + q^2 + q^3 + q^4 = (1 + q^3) + (1 + q^2 + q^4)$$

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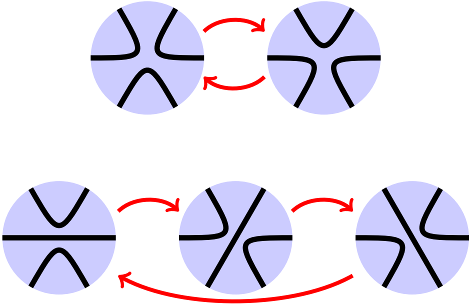

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Example: Catalan

- X is the set of non-crossing matchings on $2r$ points.
- c is the rotation map of order $2r$.
- $P(q)$ is the q-Catalan number $q^{r/(r-1)} \frac{1}{[r+1]_q} \begin{bmatrix} 2r \\ r \end{bmatrix}_q$.

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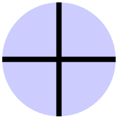
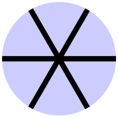
Matchings are in bijection with standard tableaux of size $2r$ such that all rows have even length (by Robinson-Schensted).
Take the polynomial P to be

$$P(q) = \sum_{\lambda \vdash r} \sum_{T:\ \mathrm{sh}(T) = (2\lambda)^t} q^{\mathrm{maj}(T)}$$

$(n+1)$-noncrossing perfect matchings

An $(n+1)$-*crossing* in a matching is a set of $n+1$ arcs such that any pair crosses.

a 2-crossing: 

a 3-crossing: 

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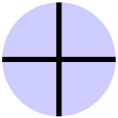
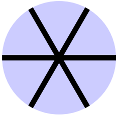
a 2-crossing: 

a 3-crossing: 

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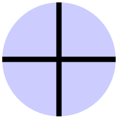
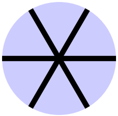
a 2-crossing:  a 3-crossing: 

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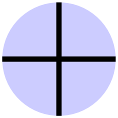
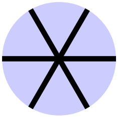
a 2-crossing: 

a 3-crossing: 

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The set of $(n+1)$-noncrossing matchings on $2r$ points is invariant under rotation.

Restriction

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Any long cycle gives an inclusion $C_r \to \mathfrak{S}_r$ and hence a restriction homomorphism

$$K(\mathfrak{S}_r) \to K(C_r) \qquad \mathrm{Symm}(r) \to \mathbb{Z}[q]/\langle q^r - 1\rangle$$

Fake degree

The restriction map is given by the fake degree, **fd**.

$$\mathbf{fd}(s_\lambda) = \sum_{T\,:\,\mathrm{sh}(T)=\lambda} q^{\mathrm{maj}(T)} \qquad \mathbf{fd}(h_\lambda) = \begin{bmatrix} r \\ \lambda \end{bmatrix}_q$$

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- The polynomial for noncrossing matchings of $2r$ is $\mathbf{fd}(s_{2^r})$.
- The polynomial for matchings of $2r$ is $\mathbf{fd}(h_r(e_2))$.

Strategy

Let U be a vector space with a linear action of $\mathfrak{S}_r$. Recall that the Frobenius character is

$$\mathbf{ch}(U) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \mathrm{tr}(\pi) p_{\lambda(\pi)}$$

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Assume U has a basis X which is permuted by the action of a long cycle, c.
Then (X, c, P) exhibits the cyclic sieving phenomenon with

$$P(q) = \mathbf{fd}\,\mathbf{ch}(U)$$

Invariant tensors

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- A computation of the Frobenius character.
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The pair (X, c) has a combinatorial construction using crystals.

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Let (V, ω) be a symplectic vector space of dimension $2n$ and $G = \mathrm{Aut}(V, \omega) = \mathrm{Sp}(2n)$.

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Theorem

The Frobenius character is

$$\mathbf{ch}\, I_{2r}(V) = \sum s_{(2\lambda)^t}$$

where the sum is over partitions of r with at most n parts.

Outline

The following statements imply that $(n+1)$-noncrossing matchings span invariant tensors:

- (Brauer) There is a natural map from matchings to invariant tensors.
- (Brauer, Weyl) The image is a spanning set.
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Theorem (Stanley, Sundaram)

The dimension of the space of invariant tensors is the number of $(n+1)$-noncrossing matchings.

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Theorem (W)

The Frobenius character of $I_r(V)$ is

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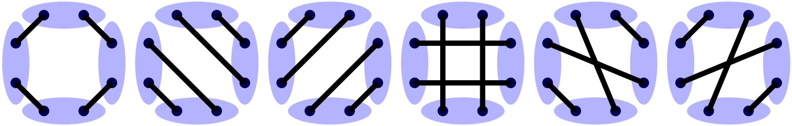
For any symmetric functions F and G,

$$\langle H(X.F(Y), G(Y)\rangle_Y = \sum_\lambda \frac{1}{z_\lambda}\langle p_\lambda(F), G\rangle p_\lambda = \sum_\lambda \langle s_\lambda(F), G\rangle s_\lambda$$

Symmetric powers

There is a basis of $I_r(V)$ with the long cycle acting by rotation indexed by the following diagrams:
Take kr points partitioned into r blocks each of size k. Take $(n+1)$-noncrossing matchings of the kr points that satisfy:

- points within a block are not connected, and
- crossing arcs have all endpoints in different blocks.



$r = 4,\ k = 2$

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The representations of the symmetric group are *different*.


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