The Cambrian Hopf Algebra

G. Châtel

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Joint work with V. Pilaud
Combinatorics

Permutations

Binary Trees

Binary Sequences

Algebra

Malvenuto-Reutenauer algebra

Loday-Ronco algebra

Solomon algebra

Geometry

FQSym = vect $\langle \mathbb{F}_\tau \mid \tau \in \mathcal{G} \rangle$

PBT = vect $\langle \mathbb{P}_T \mid T \in \mathcal{B}T \rangle$

Rec = vect $\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$

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The Cambrian Hopf Algebra
1 Combinatorics
   • Binary trees
   • Cambrian trees
   • Cambrian lattices

2 Algebra
   • FQSym
   • The Cambrian algebra
1 Combinatorics
   - Binary trees
   - Cambrian trees
   - Cambrian lattices

2 Algebra
   - FQSym
   - The Cambrian algebra
Binary trees

**Binary search tree** = directed and labeled tree such that

![Diagram of a binary search tree]

**increasing tree** = directed and labeled tree such that labels increase along arcs

**leveled binary tree** = directed tree with a binary search tree labeling and an increasing labeling
The sylvester correspondence $\Rightarrow$ permutations $\mapsto$ leveled binary trees.

Exm: permutation 6275134

```
\begin{center}
\begin{tikzpicture}

\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (2,0) {3};
\node (4) at (3,0) {4};
\node (5) at (4,0) {5};
\node (6) at (5,0) {6};
\node (7) at (6,0) {7};

\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (5);
\draw (5) -- (6);
\draw (6) -- (7);
\end{tikzpicture}
\end{center}
```
The sylvester correspondence $\rightarrow$ permutations $\mapsto$ leveled binary trees.

Exm: permutation 6275134

```
1
7
6
5
4
3
2
1
```

```
1 2 3 4 5 6 7
```
The sylvester correspondence $= \text{permutations} \longleftrightarrow \text{leveled binary trees}.$

Exm: permutation 6275134

Diagram of a leveled binary tree corresponding to the permutation 6275134.
The sylvester correspondence $\rightarrow$ permutations $\leftrightarrow$ leveled binary trees.

Exm: permutation 6275134
The sylvester correspondence $\mapsto$ permutations $\longmapsto$ leveled binary trees.

Exm: permutation 6275134

```
1 2 3 4 5 6 7
```

```
7 6 5 4 3 2 1
```

```
1 2 3 4 5 6 7
```
The sylvester correspondence $= \text{permutations} \leftrightarrow \text{leveled binary trees}$.

Exm: permutation 6275134
The sylvestre correspondence $= \text{permutations} \leftrightarrow \text{leveled binary trees}$. 

Exm: permutation 6275134
The sylvester correspondence $\Rightarrow$ permutations $\leftrightarrow$ leveled binary trees.

Exm: permutation 6275134

![Binary tree diagram with nodes labeled 1 to 7, illustrating the correspondence with a permutation.](image_url)
Cambrian trees

Cambrian tree = directed and labeled tree such that

\[ \begin{array}{c}
  \begin{array}{c}
    ? \\
    \langle j \rangle \\
    > j
  \end{array}
  \end{array} \]

increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling
Cambrian trees and triangulations

Cambrian trees are dual to triangulations of polygons

signature $\leftrightarrow$ vertices above or below $[0, 8]$
node $j$ $\leftrightarrow$ triangle $i < j < k$

For any signature $\varepsilon$, there are $C_n = \frac{1}{n+1} \binom{2n}{n} \varepsilon$-Cambrian trees
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

Exm: signed permutation $\underline{2\,7\,5\,1\,3\,4\,6}$

```
    • 3 • 6 • 7 •
  7 --
  6 ---
  5 ---
  4 ---
  3 ---
  2 ---
  1 ---

• 1 • 2 • 4 • 5 •
```
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\leftrightarrow$ leveled Cambrian tree.

Exm: signed permutation $2\overline{7}5\overline{1}3\overline{4}\overline{6}$

```
1 2 4 5
```

```
1
```

```
6 7
```

```
3
```

```
\text{Reading. Cambrian lattices. 2006}
```

```
\text{Lange-Pilaud. Associahedra via spines. 2015}
```

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The Cambrian Hopf Algebra
**Signed permutations to Cambrian trees**

*Cambrian correspondence* = signed permutation $\leftrightarrow$ leveled Cambrian tree.

**Exm:** signed permutation $2\overline{7}51\overline{3}4\overline{6}$

![Diagram showing the Cambrian correspondence from a signed permutation to a Cambrian tree.]

Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015
**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

Example: signed permutation $\underline{2\overline{7}} \underline{5\overline{1}} \underline{3\overline{4}} \overline{6}$

Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015
**Cambrian correspondence** = signed permutation $\rightarrow$ leveled Cambrian tree.

Exm: signed permutation $\begin{array}{c}2\bar{7}51\bar{3}\bar{4}\bar{6} \end{array}$

Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

*Exm:* signed permutation $\overline{2751346}$

![Diagram ofCambrian Hopf Algebra](image)

Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

**Exm:** signed permutation $2\_75134\_6$

![Diagram of a Cambrian tree corresponding to the signed permutation $2\_75134\_6$.]
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

**Exm:** signed permutation $\overline{2751346}$

![Diagram of a leveled Cambrian tree](image)

Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015
Signed permutations to Cambrian trees

**Cambrian correspondence** = signed permutation $\mapsto$ leveled Cambrian tree.

Exm: signed permutation $\bar{2751}\bar{34}\bar{6}$

\[ P(\tau) = \text{P-symbol of } \tau = \text{Cambrian tree produced by the Cambrian corresp.} \]
\[ Q(\tau) = \text{Q-symbol of } \tau = \text{increasing tree produced by the Cambrian corresp.} \]

(analogous to the Robinson-Schensted algorithm)
The Cambrian congruence

$\varepsilon$-Cambrian congruence = transitive closure of the rewriting rules

\[ \cdots ac \cdots b \cdots \equiv_{\varepsilon} \cdots ca \cdots b \cdots \quad \text{if } a < b < c \text{ and } \varepsilon_b = - \]

\[ \cdots \overline{b} \cdots ac \cdots \equiv_{\varepsilon} \cdots \overline{b} \cdots ca \cdots \quad \text{if } a < b < c \text{ and } \varepsilon_b = + \]

where $a, b, c$ are elements of $[n]$.

Proposition [reformulating Reading 2006]

\[ \tau \equiv_{\varepsilon} \tau' \iff P(\tau) = P(\tau') \]
Proposition [reformulating Reading 2006]

\[ \mathbf{P}^{-1}(T) = \mathcal{L}(T) \]

\[ T = \begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array} \]

\[ \mathbf{P}^{-1}(T) = \mathcal{L}(T) = \{2137546, 2173546, 2175346, 2713546, 2715346, 2751346, 2751346, 7213546, 7215346, 7251346, 7521346\} \]
Rotation on Cambrian trees $\leftrightarrow$ flips on triangulations.
Rotation and Cambrian lattices

Rotation operation preserves Cambrian trees:

rotation of $i \rightarrow j$

$T \rightarrow T'$

increasing rotation = rotation of edge $i \rightarrow j$ where $i < j$

Proposition [reformulating Reading 2006]

The transitive closure of the increasing rotation graph is the Cambrian lattice. $P$ defines a lattice homomorphism from the weak order to the Cambrian lattice.
Rotations and Cambrian lattices
1 Combinatorics
   • Binary trees
   • Cambrian trees
   • Cambrian lattices

2 Algebra
   • FQSym
   • The Cambrian algebra
Two products on permutations

For \( \tau \in S_n \) and \( \tau' \in S_{n'} \) with \( a \in [n] \), \( b \in [n'] \), define

- shifted concatenation \( \tau \bar{\odot} \tau' = [\tau(1), \ldots, \tau(n), \tau'(1) + n, \ldots, \tau'(n') + n] \in S_{n+n'} \)
- shifted shuffle product \( \tau \bar{\shuffle} \tau' = au \bar{\shuffle} bv = a(u \bar{\shuffle} bv) + (b + |au|)(au \bar{\shuffle} v) \)
- convolution product \( \tau \star \tau' = (\tau^{-1} \bar{\shuffle} \tau'^{-1})^{-1} \)

When we compute products of permutations, there is no multiplicities so we can consider that the output of the shuffle is a set of permutations.

\[
\begin{align*}
12 \bar{\shuffle} 231 &= \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\} \\
12 \star 231 &= \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}
\end{align*}
\]

\[
\begin{array}{c}
\text{concatenation} & \text{shuffle} & \text{convolution}
\end{array}
\]
The Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra = Hopf algebra FQSym with basis \((F_\tau)_{\tau \in \mathcal{S}}\)
and where

\[ F_\tau \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \ast \tau'} F_\tau \otimes F_{\tau'} \]

Malvenuto-Reutenauer. Duality between Quasi-Symmetric functions and the Solomon Descent Algebra. 1995
The Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra = Hopf algebra FQSym with basis \( (F_\tau)_{\tau \in \mathcal{S}} \)
and where

\[
F_\tau \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \ast \tau'} F_\tau \otimes F_{\tau'}
\]

Malvenuto-Reutenauer. Duality between Quasi-Symmetric functions and the Solomon Descent Algebra. 1995

Definition: Combinatorial Hopf Algebras

A Combinatorial Hopf Algebra = combinatorial vector space \( \mathcal{B} \) endowed with

\[
\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \\
\text{coproduct } \Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}
\]

which are “compatible”, i.e.,

\[
\Delta(f \cdot g) = \Delta(f) \cdot \Delta(g)
\]
Two products on signed permutations

For signed permutations:

\[
\begin{align*}
\bar{12} \uplus \bar{231} &= \{\bar{12}45\bar{3}, \bar{14}2\bar{5}\bar{3}, \bar{14}5\bar{2}\bar{3}, \bar{41}2\bar{5}\bar{3}, \bar{41}5\bar{2}\bar{3}, \bar{45}\bar{12}\bar{3}, \bar{45}\bar{13}\bar{2}, \bar{45}\bar{21}\bar{3}\}, \\
\bar{12} \ast \bar{231} &= \{\bar{12}45\bar{3}, \bar{13}4\bar{5}\bar{2}, \bar{14}3\bar{5}\bar{2}, \bar{15}3\bar{4}\bar{2}, \bar{23}\bar{4}\bar{5}\bar{1}, \bar{24}3\bar{5}\bar{1}, \bar{25}3\bar{4}\bar{1}, \bar{34}\bar{2}\bar{5}\bar{1}, \bar{35}\bar{2}\bar{4}\bar{1}, \bar{45}\bar{2}\bar{3}\bar{1}\}.
\end{align*}
\]

Concatenation, shuffle, convolution

Signed analog of Malvenuto-Reutenauer [Novelli, Thibon 2010]

\[\text{FQSym}_\pm = \text{Hopf algebra with basis } (\mathbb{F}_\tau)_{\tau \in \mathcal{S}_\pm} \text{ and where} \]

\[\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \uplus \tau'} \mathbb{F}_\sigma \quad \text{and} \quad \Delta \mathbb{F}_\sigma = \sum_{\sigma \in \tau \ast \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}\]
The Cambrian algebra as a subalgebra of FQSym$\pm$

**The Cambrian algebra** = subspace Camb of FQSym$\pm$ generated by

$$P_T := \sum_{\tau \in \mathcal{S}_\pm} F_\tau = \sum_{\tau \in \mathcal{L}(T)} F_\tau,$$

for all Cambrian trees $T$.

$$P = F_{\begin{array}{cccc} 2 & 1 & 3 & 7 \hline 5 & 4 & 6 \end{array}} + F_{\begin{array}{cccc} 2 & 1 & 7 & 3 \hline 5 & 4 \end{array}} + F_{\begin{array}{cccc} 2 & 1 & 7 & 5 \hline 3 & 4 \end{array}} + F_{\begin{array}{cccc} 2 & 7 & 1 & 3 \hline 5 & 4 \end{array}} + F_{\begin{array}{cccc} 2 & 7 & 1 & 5 \hline 3 & 4 \end{array}} + F_{\begin{array}{cccc} 2 & 7 & 5 & 1 \hline 3 & 4 \end{array}} + F_{\begin{array}{cccc} 7 & 2 & 1 & 3 \hline 5 & 4 \end{array}} + F_{\begin{array}{cccc} 7 & 2 & 1 & 5 \hline 3 & 4 \end{array}} + F_{\begin{array}{cccc} 7 & 5 & 2 & 1 \hline 3 & 4 \end{array}}$$

**Theorem [C.-Pilaud]**

Camb is a Hopf subalgebra of FQSym$\pm$.

(*i.e., the Cambrian congruence is “compatible” with the product and coproduct in FQSym$\pm$*)

GAME: Explain the product and coproduct directly on the Cambrian trees...
Product in the Cambrian algebra

\[ \mathbf{P} \cdot \mathbf{P} = F_{12} \cdot (F_{213} + F_{231}) \]
\[ = \left( \frac{F_{12435}}{F_{14253}} + \frac{F_{14253}}{F_{41253}} + \frac{F_{14235}}{F_{41235}} \right) + \left( \frac{F_{14325}}{F_{41325}} + \frac{F_{14325}}{F_{41325}} + \frac{F_{14352}}{F_{41352}} + \frac{F_{14352}}{F_{41352}} + \frac{F_{41352}}{F_{41532}} + \frac{F_{41352}}{F_{41532}} \right) + \left( \frac{F_{43125} + F_{43152}}{F_{43512} + F_{43512}} \right) \]
\[ = \mathbf{P} + \mathbf{P} + \mathbf{P} \]

**Proposition [C.-Pilaud]**

For any Cambrian trees \( T \) and \( T' \),

\[ \mathbf{P}_T \cdot \mathbf{P}_{T'} = \sum_{T \leq_{\text{Camb}} S \leq_{\text{Camb}} T} \mathbf{P}_S \]
Product in the Cambrian algebra

Proposition [C.-Pilaud]

For any Cambrian trees $T$ and $T'$,

$$ P_T \cdot P_{T'} = \sum_{S \leq_{\text{Camb}} T \rightarrow T'} P_S $$

\begin{itemize}
  \item For every Cambrian tree $T$, $\mathcal{L}(T)$ is an interval of the weak order.
\end{itemize}
Proposition [C.-Pilaud]

For any Cambrian trees $T$ and $T'$,

$$P_T \cdot P_{T'} = \sum_{T \leq_{\text{Camb}} S \leq_{\text{Camb}} T'} P_S$$

- For every Cambrian tree $T$, $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\sigma', \tau'\tau']$. 

Proposition [C.-Pilaud]

For any Cambrian trees $T$ and $T'$,

$$P_T \cdot P_{T'} = \sum_{S \leq \text{Camb} \leq \text{Camb}} P_S$$

- For every Cambrian tree $T$, $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\sigma', \tau'\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

$$P(\sigma\sigma') = \begin{array}{c} \text{T} \end{array} \quad \text{P}(\tau'\tau) = \begin{array}{c} \text{T'} \end{array}$$
Product in the Cambrian algebra

Proposition [C.-Pilaud]

For any Cambrian trees $T$ and $T'$,

$$P_T \cdot P_{T'} = \sum_{T \leq_{\text{Camb}} S \leq_{\text{Camb}} T'} P_S$$

- For every Cambrian tree $T$, $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma \sigma', \tau' \tau]$.
- $\mathcal{L}(T) = [\sigma, \tau]$, $\mathcal{L}(T') = [\sigma', \tau']$

$$P(\sigma \sigma') = \begin{array}{c} T' \\ T \end{array} \quad P(\tau' \tau) = \begin{array}{c} T \\ T' \end{array}$$

- Permutations in $[\sigma \sigma', \tau' \tau]$ form a union of Cambrian classes because $\text{Camb}$ is a subalgebra of $\text{FQSym}_{\pm}$. 
Proposition [C.-Pilaud]

For any Cambrian trees \( T \) and \( T' \),

\[
P_T \cdot P_{T'} = \sum_{\substack{S \leq \text{Camb} T \leq \text{Camb} \overline{T'} \leq \text{Camb} \overline{T} \leq \text{Camb} \overline{T} \leq \text{Camb} \overline{T}}} P_S
\]

- For every Cambrian tree \( T \), \( \mathcal{L}(T) \) is an interval of the weak order.
- \([\sigma, \tau] \sqcup [\sigma', \tau'] = [\sigma\sigma', \tau'\tau] \).
- \( \mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau'] \)

\[
P(\sigma\sigma') = \begin{cases} \overline{T'} & \text{if } \sigma = \tau \rightleftharpoons \text{T} \end{cases} \quad P(\tau'\tau) = \begin{cases} \overline{T'} & \text{if } \sigma = \tau \rightleftharpoons \text{T} \end{cases}
\]

- Permutations in \([\sigma\sigma', \tau'\tau]\) form a union of Cambrian classes because Camb is a subalgebra of FQSym\(\pm\).
- These trees form an interval because \( P \) is a lattice homomorphism.
Combinatorics
Algebra

FQSym
The Cambrian algebra

Coproduct in the Cambrian algebra

\[ \Delta P_\gamma = \Delta (F_{213} + F_{231}) \]

\[ = 1 \otimes (F_{213} + F_{231}) + F_{1} \otimes F_{12} + F_{1} \otimes F_{21} + F_{21} \otimes F_{1} + F_{12} \otimes F_{1} + (F_{213} + F_{231}) \otimes 1 \]

\[ = 1 \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes 1. \]

Proposition [C.-Pilaud]

For any Cambrian tree \( S \),

\[ \Delta P_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} P_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} P_{T'} \right) \]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the Cambrian forests above and below \( \gamma \) respectively.
Coproduct in the Cambrian algebra

\[
\Delta P = \Delta (F_{213} + F_{231})
\]

\[
= 1 \otimes (F_{213} + F_{231}) + F_1 \otimes F_{12} + F_1 \otimes F_{21} + F_{21} \otimes F_1 + F_{12} \otimes F_1 + (F_{213} + F_{231}) \otimes 1
\]

\[
= 1 \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes 1
\]

Proposition [C.-Pilaud]

For any Cambrian tree \( S \),

\[
\Delta P_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} P_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} P_{T'} \right)
\]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the Cambrian forests above and below \( \gamma \) respectively.
Study of multiplicative basis.
Hopf algebra on twin Cambrian trees.
Hopf algebra on Schröder-Cambrian trees.