

Combinatorics of Hecke algebras

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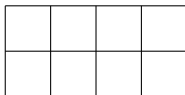
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Outline

- Introduction to Hecke algebras (partitions and crystals)
 - cores and the Mullineux map for Hecke algebras of type A
 - the modular branching rule (in the original sense)
 - the Dipper-James-Murphy conjecture for Hecke algebras of type B
 - cores and Erdmann-Nakano's theorem for Hecke algebras of type A
- Towards generalization to other affine Lie types
 - finite quiver Hecke algebras
 - cores as extremal weights of the basic module
 - Erdmann-Nakano type theorems
 - always special biserial algebras for tame type (?)

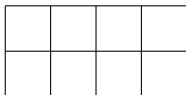
Core partitions

Everybody knows what a partition is. We visualize a partition as a Young diagram in British style.

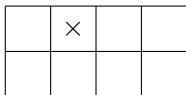


Core partitions

Everybody knows what a partition is. We visualize a partition as a Young diagram in British style.



We choose a node in the diagram.



Then, we may consider the hook which the node defines. If there is no hook of length e , we say that the partition is an **e -core**.

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$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ (if } j \neq i \pm 1), \quad s_i s_j s_i = s_j s_i s_j \text{ (if } j = i \pm 1).$$

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Let F be a field of positive characteristic ℓ . The relations also give the defining relations of the group algebra FS_n . The group algebra belongs to a one-parameter family of algebras $H_n(q)$ ($q \in F^\times$), defined by

$$(s_i - q)(s_i + 1) = 0, \quad s_i s_j = s_j s_i \text{ (if } j \neq i \pm 1\text{)}, \quad s_i s_j s_i = s_j s_i s_j \text{ (if } j = i \pm 1\text{)}.$$

$H_n(q)$ are called Hecke algebras of type A . We often start with a field of characteristic 0 and $q \neq 1$ an ℓ^{th} root of unity. Then we reduce it modulo ℓ to obtain FS_n . Thus, it suffices to consider $H_n(q)$ only.

Block algebras (general definition)

If A is a finite dimensional algebra, we may decompose it into a direct sum

$$A = A_1 \oplus \cdots \oplus A_s$$

of two sided ideals A_i . Write $1 = \sum e_i$ along the decomposition. Then, we have $e_i e_j = \delta_{ij} e_i$, and $ae_i = e_i a$, for $a \in A$. If A_i is no more decomposable into a direct sum of two sided ideals, we call A_i a **block algebra** of A . We view A_i as an algebra with unit e_i .

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Example 2.1

Let $A = FS_3$ and suppose that 2 and 3 are invertible in F . We define

$$e_1 = \frac{1}{6} \sum_{\sigma \in S_3} \sigma, \quad e_2 = \frac{1}{6} \sum_{\sigma \in S_3} \text{sgn}(\sigma) \sigma, \quad e_3 = 1 - e_1 - e_2.$$

Then, $Ae_1 \simeq \text{Mat}(1, F)$, $Ae_2 \simeq \text{Mat}(1, F)$ and $Ae_3 \simeq \text{Mat}(2, F)$ are block algebras. Thus, $A = A_1 \oplus A_2 \oplus A_3$ if we set $A_i = Ae_i = e_i A$.

Block algebras and e -cores for $A = FS_3$

Let us continue the previous example. Observe that

- The projection $A = FS_3 \rightarrow A_1 = Fe_1$, where

$$e_1 = \frac{1}{6} \sum_{\sigma \in S_3} \sigma$$

gives the trivial representation of S_3 , as $ge_1 = e_1$, for $g \in S_3$. We label the trivial representation with the partition (3) .

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- The projection $A = FS_3 \rightarrow A_2 = Fe_2$, where

$$e_2 = \frac{1}{6} \sum_{\sigma \in S_3} \text{sgn}(\sigma)\sigma$$

gives the sign representation of S_3 , as $ge_2 = \text{sgn}(g)e_2$, for $g \in S_3$. We label the sign representation with the partition (1^3) .

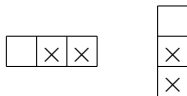
- The remaining $A = FS_3 \rightarrow A_3$ gives the representation of S_3 which we label with the partition $(2, 1)$.

Block algebras and e -cores for $A = FS_3$ (cont'd)

- If $\text{char}(F) = 2$, we may not define e_1 and e_2 individually, but we may still well-define its sum $e_1 + e_2$. Thus, A has two block algebras.

$$e_1 + e_2 = \frac{1}{3}(1 + s_1s_2 + s_2s_1), \quad e_3 = \frac{1}{3}(2 - s_1s_2 - s_2s_1).$$

How to merge idempotents may be controlled by 2-cores: observe that we may remove a 2-hook from (3) and (1^3) to share the 2-core (1) , while $(2, 1)$ is a 2-core.



- If $\text{char}(F) = 3$, we may not define any of e_1 , e_2 , e_3 , $e_1 + e_2$, $e_1 + e_3$ and $e_2 + e_3$. Thus, A has the unique block algebra, namely A itself. Now, observe that we may remove a 3-hook from all of (3) , $(2, 1)$ and (1^3) to obtain the common 3-core \emptyset .

Labeling block algebras with e -cores

The e -core of a partition is the partition which is obtained by successive removal of e -hooks.

Labeling block algebras with e -cores

The e -core of a partition is the partition which is obtained by successive removal of e -hooks. Next is the precise statement for the labeling.

Proposition 2.2

Suppose that e is a *prime*. For each partition $\lambda \vdash n$, we have the corresponding irreducible character χ_λ of S_n . We define

$$e_\lambda = \frac{\chi_\lambda(1)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma^{-1}) \sigma.$$

Let κ be an e -core, and \mathcal{P}_κ the set of all partitions which have κ as their common e -core. Then, $e_\kappa = \sum_{\lambda \in \mathcal{P}_\kappa} e_\lambda$ is well-defined, and $B_\kappa = Ae_\kappa$ is a *block algebra of FS_n* . Any block algebra is obtained in this way.

We may define *block algebras B_κ of $H_n(q)$* in the similar manner, but for *arbitrary* $e \in \mathbb{N}$.

Specht modules for $H_n(q)$

Let λ be a partition of n , S_λ the corresponding Young subgroup, which is generated by $s_i = (i, i + 1)$ with $i \neq \lambda_1, \lambda_1 + \lambda_2, \dots$. Define

$$m_\lambda = \sum_{w \in S_\lambda} w.$$

Then, multiplying m_λ with certain distinguished right and left coset representatives on both sides, we obtain the **Murphy basis**

$$\{m_{st} \mid \lambda \vdash n, s, t \in \text{ST}(\lambda)\},$$

for FS_n . The same construction gives the **Murphy basis for $H_n(q)$** .

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for FS_n . The same construction gives the **Murphy basis for $H_n(q)$** . It gives a filtration of $H_n(q)$ by two-sided ideals such that each of the successive quotients is isomorphic to $S^\lambda \otimes {}^t S^\lambda$, for some module S^λ , and its “transpose” ${}^t S^\lambda$. The modules S^λ are called **Specht modules**.

Labeling irreducible modules with restricted partitions

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Theorem 2.4

Let $e = \min\{k \mid 1 + q + \dots + q^{k-1} = 0\}$. If λ is *e-restricted*, then the $H_n(q)$ -module S^λ has unique irreducible quotient. We denote it by D^λ , namely $D^\lambda = \text{cosoc}(S^\lambda)$. Then, the set

$$\{D^\lambda \mid \lambda \text{ is } e\text{-restricted.}\}$$

is a complete set of isomorphism classes of irreducible $H_n(q)$ -modules.

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Our aim to generalize the results and expand the realm of algebraic combinatorics. It has turned out that Kashiwara crystal is natural language for the purpose. For example, let us consider the rule

$$S^\lambda \otimes \text{sgn} \simeq S^{\lambda'},$$

where λ' is the conjugate partition of λ .

Remark 2.5

Let $\rho_\lambda : H_n(q) \rightarrow \text{End}(S^\lambda)$ be the action. Considering the automorphism of $H_n(q)$ defined by $\theta : s_i \mapsto -qs_i^{-1}$, we may interpret $S^\lambda \otimes \text{sgn}$ as the module S^λ with the new action given by $\rho_\lambda \circ \theta$. Moreover, we have the same rule for $H_n(q)$ as above for generic q .

The Mullineux map

As FS_n is $H_n(q)$ with $q = 1$, let us consider $H_n(q)$ from the beginning. Then, the condition that $\text{char}(F)$ does not divide $n!$ is $P_n(q) \neq 0$, for

$$P_n(t) = (1 + t)(1 + t + t^2) \cdots (1 + t + \cdots + t^{n-1}).$$

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Mullineux conjectured a rule to compute $m(\lambda)$ from λ in 1979, and Kleshchev proved the rule in 1996. However, conjugate partitions do not appear in their description of the map $\lambda \mapsto m(\lambda)$.

Language of Kashiwara crystals

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Let \mathfrak{g} be a Kac-Moody Lie algebra. $(B, \text{wt}, \tilde{e}_i, \tilde{f}_i)$ is a semi-normal \mathfrak{g} -crystal if B is a set, $\text{wt} : B \rightarrow P$ is a map to the weight lattice P of \mathfrak{g} , \tilde{e}_i and \tilde{f}_i are maps $B \rightarrow B \sqcup \{0\}$, such that

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$$\epsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\},$$

then the following hold.

- (1) $\epsilon_i, \varphi_i : B \rightarrow \mathbb{Z}$ and $\varphi_i(b) = \epsilon_i(b) + \langle \text{wt}(b), \alpha_i^\vee \rangle$.
- (2) If $\tilde{e}_i b \neq 0$ then $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\epsilon_i(\tilde{e}_i b) = \epsilon_i(b) - 1$,
 $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
- (3) If $\tilde{f}_i b \neq 0$ then $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$, $\epsilon_i(\tilde{f}_i b) = \epsilon_i(b) + 1$,
 $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.
- (4) For $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$.

Misra-Miwa model

Let \mathfrak{g} be of affine type $A_\ell^{(1)}$. Then, we choose the weight lattice as

$$P = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_\ell \oplus \mathbb{Z}\delta,$$

where Λ_i are the fundamental weights and δ is the null root.

Theorem 2.6

Let $e = \ell + 1$. Then, $B(\Lambda_0)$ is realized on the set of e -restricted partitions. The weight of an e -restricted partition λ is

$$\text{wt}(\lambda) = \Lambda_0 - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \#\{(j, k) \in \lambda \mid -j + k \equiv i \pmod{e}\} \alpha_i,$$

and the map \tilde{e}_i and \tilde{f}_i are defined by the signature rule.

Littelmann path model

Another realization of $A_{e-1}^{(1)}$ -crystal $B(\Lambda_0)$ is given by a version of the Littelmann path model. We explain Kashiwara's treatment of the path model in our setting.

Definition 2.7

Let B and B' be crystals. We call a map $\psi : B \rightarrow B'$ *crystal morphism of amplitude h* if

- (i) $wt(\psi(b)) = hwt(b)$, $\epsilon_i(\psi(b)) = h\epsilon_i(b)$ and $\varphi_i(\psi(b)) = h\varphi_i(b)$,
- (ii) $\psi(\tilde{e}_i b) = \tilde{e}_i^h \psi(b)$ and $\psi(\tilde{f}_i b) = \tilde{f}_i^h \psi(b)$, for all $b \in B$.

Proposition 2.8

Let Λ be dominant integral. Then there exists a unique crystal morphism $S_h : B(\Lambda) \rightarrow B(h\Lambda)$ of amplitude h , for all $h \in \mathbb{N}$.

Littelmann path model (cont'd)

Let us start with the Misra-Miwa model and we identify $B(\Lambda_0)$ with the set of e -restricted partitions. Let $\lambda \in B(\Lambda_0)$ be an e -restricted partition.

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$$S_h(\lambda) = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(h)}.$$

We denote it by $S_h(\lambda)^{1/h} = \lambda^{(1)\otimes 1/h} \otimes \cdots \otimes \lambda^{(h)\otimes 1/h}$, and replace $(\mu^{\otimes 1/h})^{\otimes k}$ with $\mu^{\otimes k/h}$, for any μ that appears in $\lambda^{(1)}, \dots, \lambda^{(h)}$.

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In this way, we may write

$$S_h(\lambda)^{1/h} = \nu_1^{\otimes a_1} \otimes \nu_2^{\otimes (a_2 - a_1)} \otimes \cdots \otimes \nu_s^{\otimes (1 - a_{s-1})},$$

where $a_0 = 0 < a_1 < \cdots < a_s = 1$ are rational integers and ν_1, \dots, ν_s are pairwise distinct e -restricted partitions.

Littelmann path model (cont'd)

Theorem 2.9

If h is sufficiently divisible then

- (1) ν_j are e -cores.
- (2) a_j and ν_j all stabilize.

Remark 2.10

Given sufficiently divisible h , we write

$$S_h(\lambda)^{1/h} = \nu_1^{\otimes a_1} \otimes \nu_2^{\otimes (a_2 - a_1)} \otimes \dots \otimes \nu_s^{\otimes (1 - a_{s-1})}$$

as above, and define π_λ to be the path $(\text{wt}(\nu_1), \dots, \text{wt}(\nu_s); a_0, \dots, a_s)$. Then π_λ is a LS-path. Let $\mathbb{B}(\Lambda_0)$ be the Littelmann path model, i.e. the crystal of those LS paths. Then, the map $B(\Lambda_0) \rightarrow \mathbb{B}(\Lambda_0)$ defined by $\lambda \mapsto \pi_\lambda$ is an isomorphism of crystals.

Kashiwara crystal for the Mullineaux map

We have explained that the Kashiwara crystal $B(\Lambda_0)$ may be realized on two combinatorial models.

- the set of e -restricted partitions.
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Now, I can explain my description of the Mullineaux map, which naturally generalizes the map $\lambda \mapsto \lambda'$:

Theorem 2.11

The Mullineaux map is given by conjugating all the e -cores ν_j :

$$(\nu_1, \dots, \nu_s; a_0, \dots, a_s) \mapsto (\nu'_1, \dots, \nu'_s; a_0, \dots, a_s).$$

If $D^\lambda = S^\lambda$, we have $s = 1$ and the rule is reduced to the original one.

Modular branching rule as another example

We continue identifying $B(\Lambda_0)$ with the set of e -restricted partitions. As Leclerc pointed out, the Brundan-Kleshchev modular branching rule may be stated as

$$\text{Soc}(\text{Res}_{H_{n-1}(q)}^{H_n(q)} D^\lambda) = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} D^{\tilde{e}_i \lambda}.$$

Remark 2.12

We introduced the notion of cyclotomic quotient in early 90's. Further, we proved modular branching rules for cyclotomic Hecke algebras and the affine Hecke algebra of type A . The rules have the common form that $\text{Soc}(\text{Res} D^\lambda)$ is the direct sum of $D^{\tilde{e}_i \lambda}$'s. Then, with Jacon and Lecouvey, we showed that the two modular branching rules are compatible with the embedding of crystals $B(\Lambda) \hookrightarrow B(\infty) \otimes T_\Lambda$.

Dipper-James-Murphy conjecture

I give yet another example where crystal theory plays a role. Let $H_n(q, Q)$ be the Hecke algebras of type B_n . It is the algebra defined by generators s_0, s_1, \dots, s_{n-1} and the relations

$$(s_0 - Q)(s_0 + 1) = 0, \quad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \quad s_0 s_i = s_i s_0 \quad (i \geq 2)$$

and the same relations among s_1, \dots, s_{n-1} as $H_n(q)$.

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To explain the Dipper-James-Murphy conjecture, we reconsider the condition that a partition λ is e -restricted, in different way.

Dipper-James-Murphy conjecture

I give yet another example where crystal theory plays a role. Let $H_n(q, Q)$ be the Hecke algebras of type B_n . It is the algebra defined by generators s_0, s_1, \dots, s_{n-1} and the relations

$$(s_0 - Q)(s_0 + 1) = 0, \quad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \quad s_0 s_i = s_i s_0 \quad (i \geq 2)$$

and the same relations among s_1, \dots, s_{n-1} as $H_n(q)$.

To explain the Dipper-James-Murphy conjecture, we reconsider the condition that a partition λ is e -restricted, in different way. Let t be a standard tableau of shape λ and assume that $1, \dots, n$ are located on $(i_1, j_1), \dots, (i_n, j_n)$, respectively. Then, we call

$$(-i_1 + j_1, \dots, -i_n + j_n) \in (\mathbb{Z}/e\mathbb{Z})^n$$

the weight of t .

Dipper-James-Murphy conjecture (cont'd)

Then, we observe that λ is e -restricted if and only if there is a standard tableau t of shape λ such that the weight of t does not coincide with any weight of tableaux of shape $\mu \triangleleft \lambda$.

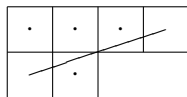
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To see this, we use the ladder decomposition of a partition. Each ladder consists of $(i, j), (i - 1, j + e - 1), (i - 2, j + 2e - 2), \dots$

Example 1

The following is an example for $e = 4$. One of the weights which does not appear as weights of tableaux of shape $\mu \triangleleft \lambda = (4, 2)$ is $(0, 1, 2, 3, 3, 0)$.



Dipper-James-Murphy conjecture (cont'd)

Dipper, James and Murphy considered the same condition for bipartitions λ that there is a standard bitableau of shape λ such that its weight does not coincide with any weight of bitableaux of shape $\mu \triangleleft \lambda$, and call such bipartitions (Q, e) -restricted bipartitions.

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Then, they conjectured in 1995 as follows.

Suppose that $-Q \in q^{\mathbb{Z}}$. Then,

$$\{D^\lambda \mid \lambda \text{ is } (Q, e)\text{-restricted.}\}$$

is a complete set of isomorphism classes of irreducible $H_n(q, Q)$ -modules.

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Remark 2.13

We published a proof of the conjecture in 2007. We used combinatorics of Fock spaces, Demazure crystals etc. in the proof.

e -cores and Erdmann-Nakano's theorem

Now, we return to e -cores. Recall that block algebras of $H_n(q)$ are labeled by e -cores κ . Erdmann and Nakano showed that $k = (n - |\kappa|)/e$ controls its representation type. As you know, a finite dimensional algebra has

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- **finite representation type** if it is not semisimple but has finitely many isomorphism classes of indecomposable modules.
- **tame representation type** if for each positive integer d , there are finitely many one parameter families to cover all the indecomposable modules of dimension d , up to isomorphisms.
- **wild representation type** if there is a functor which embeds the category of indecomposable modules over the free algebra in two variables to the module category of the algebra. In other words, the functor respects indecomposability and isomorphism classes.

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It tells us when we may expect a kind of Jordan normal form.

Finite quiver Hecke algebras

Now, it is time for departing affine type $A_\ell^{(1)}$. Khovanov and Lauda categorified the negative half of the quantum group by using affine quiver Hecke algebras, or Khovanov-Lauda-Rouquier algebras, in their paper “A diagrammatic approach to categorification of quantum groups I”, and proposed the study of the cyclotomic quotient of the affine quiver Hecke algebras.

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The conjecture itself was proved by Kang and Kashiwara. Let β be a non-negative linear combination of simple roots α_i . We denote their $R(\beta; \Lambda_0)$ by $R^{\Lambda_0}(\beta)$ and call them **finite quiver Hecke algebras**.

Categorification picture

We consider categorification of the basic module $V(\Lambda_0)$ over an affine Kac-Moody Lie algebra. That is, we have

$$\bigoplus_{\beta \geq 0} K_0(R^{\Lambda_0}(\beta)\text{-mod}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq V(\Lambda_0),$$

where $\beta \geq 0$ means $\beta \in \mathbb{Z}_{\geq 0}\alpha_0 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_\ell$. This generalizes the Frobenius characteristic map, which is for affine type A_∞ .

Remark 3.1

The direct sum $R^{\Lambda_0}(n)$ of $R^{\Lambda_0}(\beta)$ over $\text{ht}(\beta) = n$ is an analogue of $H_n(q)$.

In this picture, the Mullineaux map is to describe the effect of an Dynkin automorphism on the labelling of irreducible $R^\Lambda(n)$ -modules in the case when Λ is fixed under the Dynkin automorphism, etc.

Lemma 12.6 in Victor Kac's book

If B is a normal \mathfrak{g} -crystal, the Weyl group W acts on B as follows.

$$s_i b = \begin{cases} \tilde{f}_i^{\langle \alpha_i^\vee, \text{wt}(b) \rangle} b, & (\text{if } \langle \alpha_i^\vee, \text{wt}(b) \rangle \geq 0.) \\ \tilde{e}_i^{-\langle \alpha_i^\vee, \text{wt}(b) \rangle} b, & (\text{if } \langle \alpha_i^\vee, \text{wt}(b) \rangle \leq 0.) \end{cases}$$

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In affine type $A_{e-1}^{(1)}$, $B(\Lambda_0)$ is realized on the set of e -restricted partitions. Then, in the categorification picture, we have that e -cores are in bijection with extremal weights of $B(\Lambda_0)$ and they form the W -orbit through the highest weight element.

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It allows us to generalize the notion of e -cores. Let \mathfrak{g} be of type

$$A_\ell^{(1)}, D_{\ell \geq 4}^{(1)}, E_{6,7,8}^{(1)}, A_{2\ell}^{(2)}, A_{2\ell-1 \geq 5}^{(2)}, D_{\ell+1 \geq 3}^{(2)}, E_6^{(2)}, D_4^{(3)}.$$

Then, the set of maximal weights $\{\kappa \mid B(\Lambda_0)_\kappa \neq \emptyset, B(\Lambda_0)_{\kappa+\delta} = \emptyset\}$ coincides with the W -orbit through the highest weight element.

Generalizing e -cores

We have noticed that if we realize $B(\Lambda_0)$ on the set of e -restricted partitions, the set of e -cores coincides with the set of maximal weights $\{\kappa \mid B(\Lambda_0)_\kappa \neq \emptyset, B(\Lambda_0)_{\kappa+\delta} = \emptyset\}$. In this case, block algebras of $H_n(q)$ are $R^{\Lambda_0}(\beta)$'s. Thus, we may not only generalize cores but the previous story to other affine types, by using the finite quiver Hecke algebras.

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Definition 3.2

Let β be a non-negative linear combination of simple roots α_j . Then, we label the algebra $R^{\Lambda_0}(\beta)$ with the maximal weight κ determined by

$$\Lambda_0 - \beta \in \kappa - \mathbb{Z}_{\geq 0}\delta.$$

*We call κ the **core weight** of β .*

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We will see that $k \geq 0$ given by $\Lambda_0 - \beta = \kappa - k\delta$ controls the representation type of $R^{\Lambda_0}(\beta)$ labeled by κ .

The core weights and the Erdmann-Nakano theorem

Recall that representation type tells us when we may study algebras and their modules in detail.

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Thus, in past several years, we studied representation type for other affine types. This part is joint work with Euiyong Park. In the work, we studied affine types with two double arrows on both ends, as well as $A_\ell^{(1)}$.

$$A_{2\ell}^{(2)} : \circ \leftarrow \cdots \leftarrow \circ \quad D_{\ell+1}^{(2)} : \circ \leftarrow \cdots \Rightarrow \circ$$

$$C_\ell^{(1)} : \circ \Rightarrow \cdots \leftarrow \circ$$

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Now we recall the definition of special biserial algebra. It is a combinatorial notion.

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- (a1) For each vertex i , the number of incoming arrows is at most 2.
- (a2) For each vertex i , the number of outgoing arrows is at most 2.
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Remark 3.3

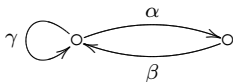
We may also determine the Auslander-Reiten quiver.

Special biserial algebras -example-

Let me give an example. We start with an algebra, and general recipe is that we manage to determine the radical series of indecomposable projective modules. Then, we may write the quiver and the relations.

Example 2

Suppose that we have obtained the following quiver and the defining relations $\beta\alpha = 0, \gamma\alpha\beta = \alpha\beta\gamma, \gamma^2 = 0$.



Each vertex has at most two incoming and outgoing arrows. If we consider a walk of length 2 which starts or ends with α , we know that it is unique. The same holds for β or γ . Thus, the algebra is special biserial.

Erdmann-Nakano type theorems

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Let $R^{\Lambda_0}(\beta)$ be of affine type $D_{\ell+1}^{(2)}$ ($\ell \geq 2$), and κ its core weight. Define $k \in \mathbb{Z}_{\geq 0}$ by $\Lambda_0 - \beta = \kappa - k\delta$. Then, $R^{\Lambda_0}(\beta)$ is

- ① simple if $k = 0$,
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Derived equivalence plays a key role in the proof, and Rouquier's derived equivalence, which categorifies the Weyl group action, is decomposed into composition of those derived equivalences coming from mutation of Brauer graph if $k = 2$. If $k = 0, 1$ the algebras are much simpler.

Summary

- We have explained that non-semisimple representation theory of Hecke algebras and their generalizations may be part of algebraic combinatorics, and Kashiwara crystal serves well as a language for statements and proofs.

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- The notion of e -cores plays important roles when we label block algebras and tell how complex the algebras are by the representation type. Further, it may be generalized naturally in Lie theoretic terms, and the representation type may be judged by Erdmann-Nakano type theorems that use generalization of e -cores.

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- The notion of e -cores plays important roles when we label block algebras and tell how complex the algebras are by the representation type. Further, it may be generalized naturally in Lie theoretic terms, and the representation type may be judged by Erdmann-Nakano type theorems that use generalization of e -cores.
- When the representation type is tame, the algebra should be a Brauer graph algebra, an object with combinatorial flavor.

Thank you for your attention.