Equivariant K-theory of Grassmannians

Oliver Pechenik

University of Illinois at Urbana-Champaign

FPSAC, Daejeon Korea
July 2015

Joint with Alexander Yong

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- Consider the Grassmannian $X = \mathrm{Gr}_k(\mathbb{C}^n)$ of k-dimensional subspaces of $\mathbb{C}^n$

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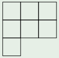
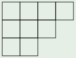
- $$c^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$$

Theorem (Littlewood-Richardson '34, Schützenberger '70s)

$c^{\nu}_{\lambda,\mu} = \#$*semistandard ballot tableaux of shape* ν/λ *and content* μ.

Example

For $\lambda =$ , $\mu =$ , $\nu =$ ,

the tableau

		1	1
1	2	2	
2	3		

witnesses $c^{\nu}_{\lambda,\mu} \geq 1$.

Cohomology theories

$H^\star(X)$ *Schur polynomials*

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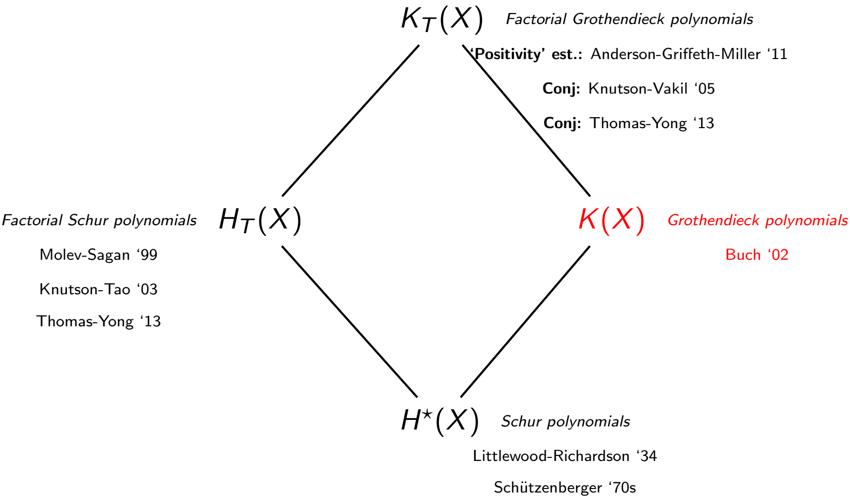
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Theorem (P.–Yong)

$k^{\nu}_{\lambda,\mu} = (-1)^{|\nu|-|\lambda|-|\mu|} \cdot$ *#semistandard ballot genomic tableaux of shape* ν/λ *and content* μ.

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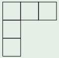
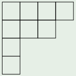
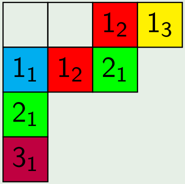
For $\lambda =$ , $\mu =$ , $\nu =$ ,

the tableau 

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Equivariant K-theory

Theorem (P.–Yong)

$K^{\nu}_{\lambda,\mu} = \sum_T \texttt{wt}\, T$, *where the sum is over all edge-labeled semistandard ballot genomic tableaux of shape ν/λ and content μ.*

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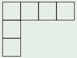
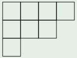
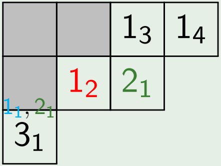
For $\lambda =$ , $\mu =$ , $\nu =$ , the tableau



contributes $-\frac{t_4}{t_7}\left(1 - \frac{t_2}{t_7}\right)\left(1 - \frac{t_2}{t_3}\right)$ to $K^{\nu}_{\lambda,\mu}$.

- By Chevalley:

$$K^{\nu}_{\lambda,\mu} = \frac{\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K^{\nu}_{\rho,\mu} - \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K^{\delta}_{\lambda,\mu} \, \mathtt{wt}(\delta/\lambda)}{(1 - \mathtt{wt}(\nu/\lambda))}$$

Outline of K_T Proof

- By Chevalley:

$$\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K^{\nu}_{\rho,\mu} = K^{\nu}_{\lambda,\mu}(1 - \mathtt{wt}(\nu/\lambda)) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K^{\delta}_{\lambda,\mu}\, \mathtt{wt}(\delta/\lambda)$$

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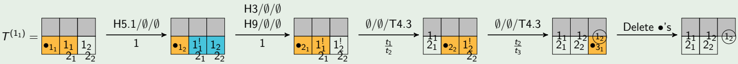

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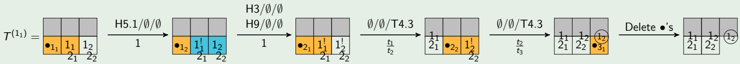
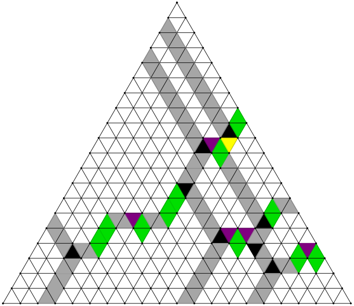
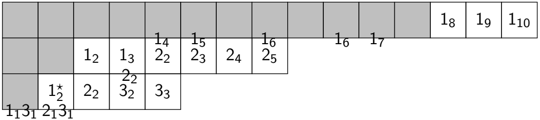
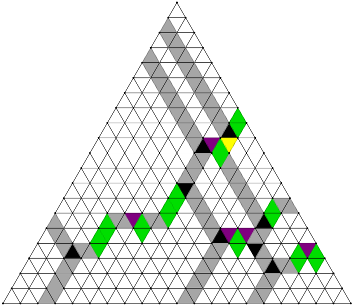


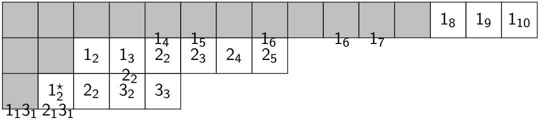
- The *jeu de taquin* is NOT weight-preserving on individual tableaux. To show that it IS weight-preserving on LHS, we develop some 'sign-reversing involutions.'

Modified Knutson-Vakil puzzles








1_8 1_9 1_{10}
1_4 2_2
1_5 2_3
1_6 2_5
1_6 1_7
1_2 1_3 2_4
$1_2^\star$ 2_2 2_2 3_2 3_3
$1_1 3_1$ $2_1 3_1$

THANK YOU!