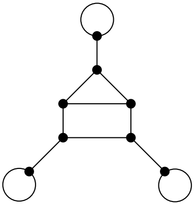
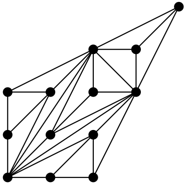
TROPICAL PLANE CURVES

Bernd Sturmfels
UC Berkeley

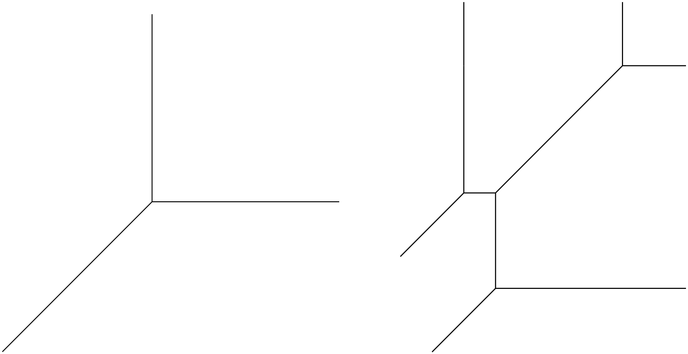




Joint work with Sarah Brodsky,
Michael Joswig, and Ralph Morrison

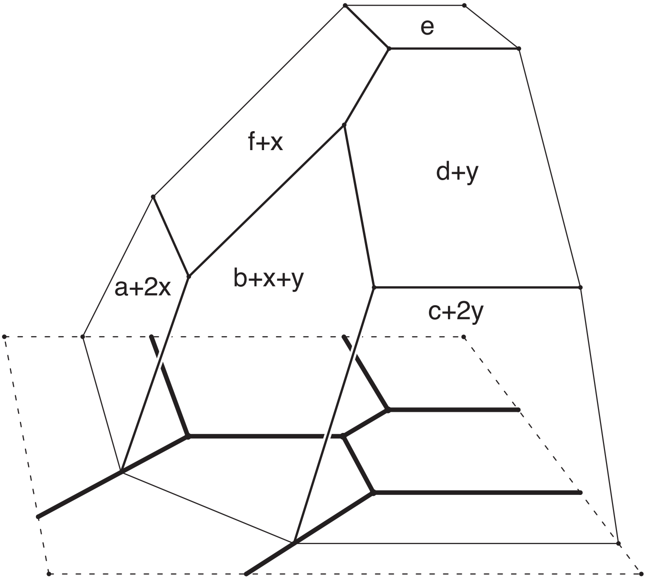
What is a Tropical Curve?

Answer 1: A *tropical plane curve* is defined by a polynomial $p(x, y)$ over the min-plus algebra.



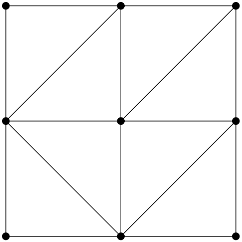
Diane Maclagan and Bernd Sturmfels:
Introduction to Tropical Geometry,
Graduate Studies in Mathematics, Vol 161,
American Mathematical Society, 2015.

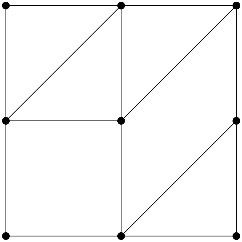
Quadric



$$p(x,y) \quad = \quad a \odot x^2 \ \oplus \ b \odot xy \ \oplus \ c \odot y^2 \ \oplus \ d \odot y \ \oplus \ e \ \oplus \ f \odot x.$$

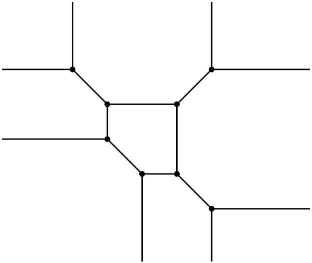
Newton Polygons

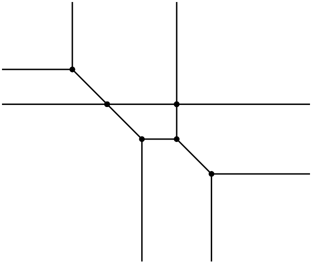




Elliptic Curve

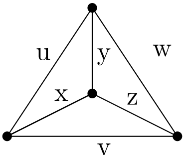
Singular Curve

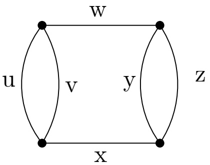


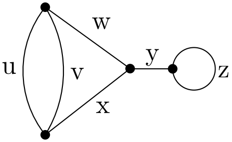


What is a Tropical Curve?

Answer 2: A *tropical curve* is a metric graph. After modifications, this graph has no 1-valent and 2-valent vertices. It has a genus g.

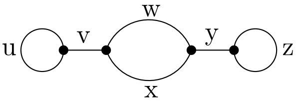


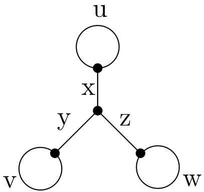




Grigory Mikhalkin and Johannes Rau:
Tropical geometry, Book in preparation.

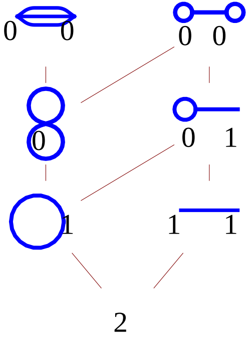
genus $g = 3$

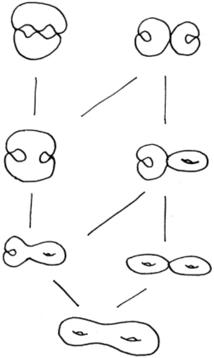


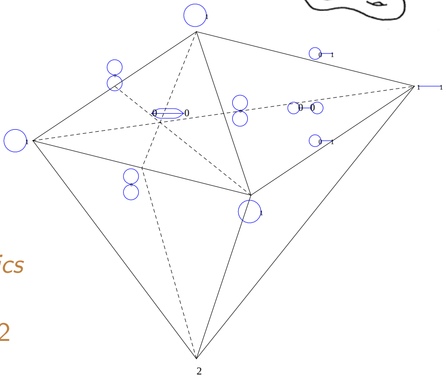


Abramovich, Baker, Caporaso, Gathmann, Gross, Markwig, Payne, . . .

Genus 2

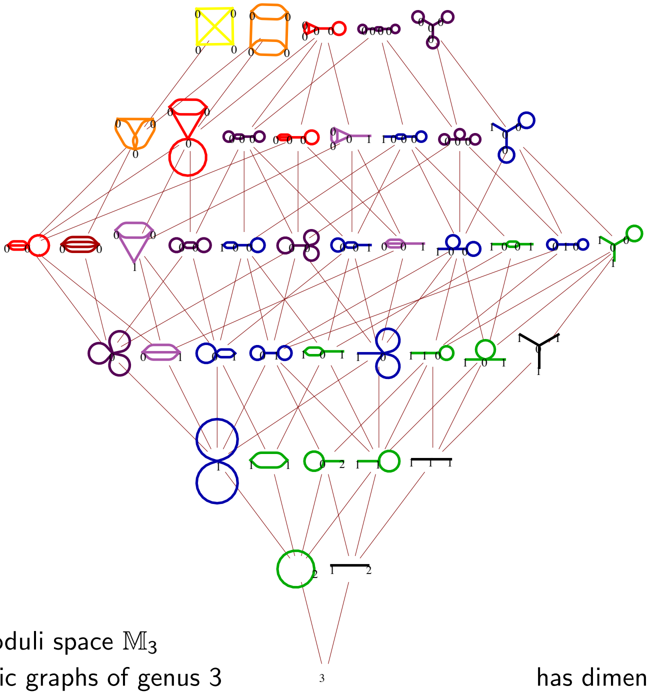






Melody Chan: *Combinatorics of the tropical Torelli map*, Algebra & NumberTh, 2012

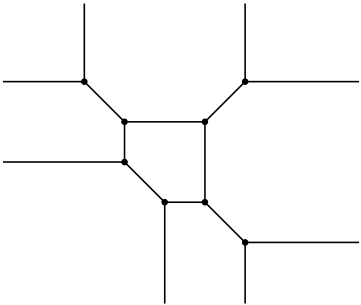
Genus 3

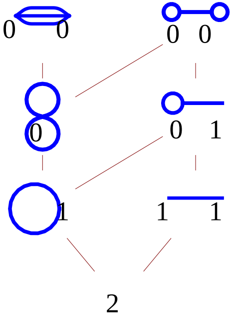


The moduli space $\mathbb{M}_3$
of metric graphs of genus 3 has dimension 6.

Questions

How are the two concepts of tropical curve related ?

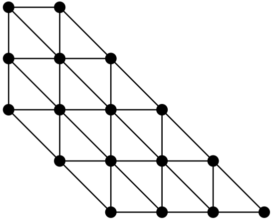


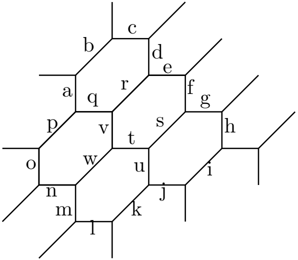


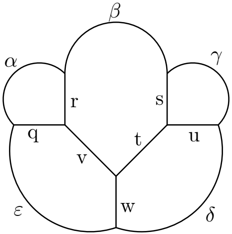
How to get from a tropical plane curve to a metric graph?

Which metric graphs arise from tropical plane curves?

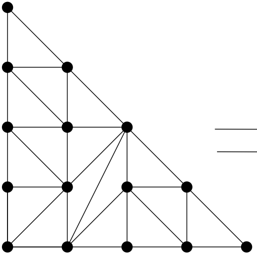
Connections

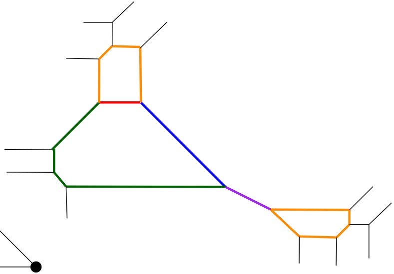


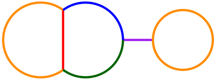




Tropical curves are dual to triangulations of convex polygons in $\mathbb{R}^2$.
Each tropical curve has a skeleton, which is a metric graph.







Dimension

The moduli space $\mathbb{M}_g$ of metric graphs of genus g has dimension $3g-3$.

Theorem

For all $g \geq 2$ there is a lattice polygon P with $g = |\mathrm{int}(P) \cap \mathbb{Z}^2|$ such that the subspace $\mathbb{M}_P$ of curves with Newton polygon P has the dimension expected from classical algebraic geometry, i.e.

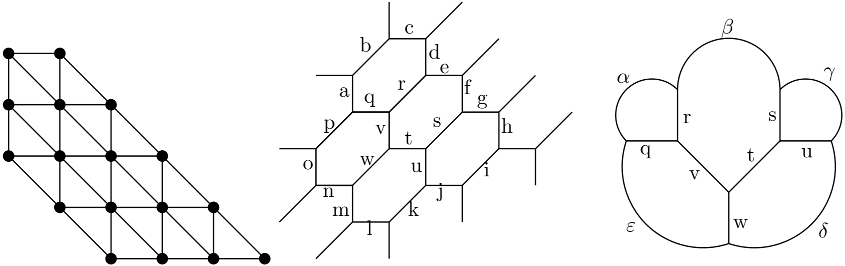
$$\dim(\mathbb{M}_g^{\mathrm{planar}}) \quad = \quad \dim(\mathbb{M}_P) \quad = \quad \begin{cases} 3 & \text{if} \quad g = 2, \\ 6 & \text{if} \quad g = 3, \\ 16 & \text{if} \quad g = 7, \\ 2g+1 & \text{otherwise.} \end{cases}$$

The cone $\mathbb{M}_\Delta$ of honeycomb curves on P attains this dimension.

The classical result is due to W. Castryck and J. Voight: *On nondegeneracy of curves*, Algebra & NumberTh, 2009

Honeycombs

The cone $\mathbb{M}_\Delta$ of honeycomb curves has the expected dimension:



Nodal sextics are plane curves of genus $g = 5$.

Their moduli dimension is $\dim(\mathbb{M}_5^{\text{planar}}) = 2g + 1 = 11$.

The ambient space $\mathbb{M}_5$ has dimension $3g - 3 = 12$.

Riddle: Find the unique relation among the 12 edge lengths.

Drawing Polygons

Proposition

For every $g \geq 1$, there are only finitely many lattice polygons P with g interior lattice points, up to integer affine isomorphisms.

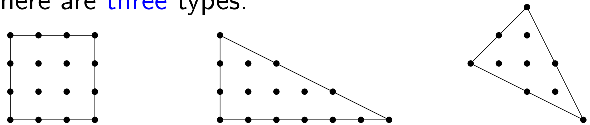
Lemma

If $P \subseteq Q$ are lattice polygons with $P_{\mathrm{int}} = Q_{\mathrm{int}}$ then $\mathbb{M}_P \subseteq \mathbb{M}_Q$.

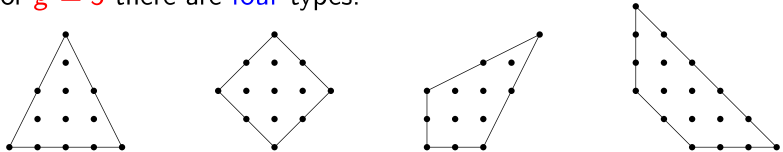
Here are all maximal polygons P with $\dim(P_{\mathrm{int}}) = 2$:

For $g = 3$, there is only one type: $T_4 = \mathrm{conv}\{(0,0),(0,4),(4,0)\}$.

For $g = 4$ there are three types:

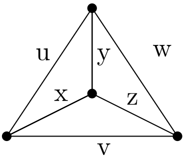


For $g = 5$ there are four types:

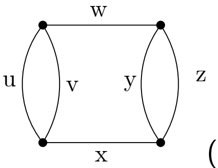


Genus 3: Graphs

$$\mathbb{M}_3^{\text{planar}} \quad = \quad \mathbb{M}_{T_4} \cup \text{ hyperelliptic curves}$$



(000)



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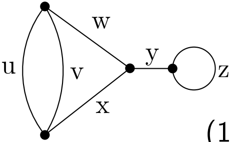
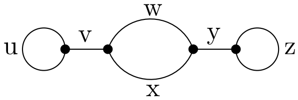
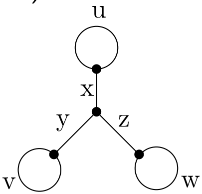


(111)



(212)



(303)

Genus 3: Results

Theorem

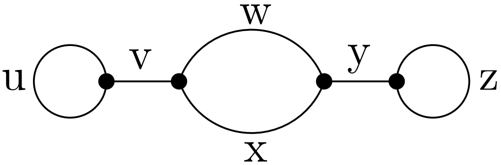
A graph arises from a smooth tropical quartic if and only if it is one of the first four, with edge lengths satisfying the following up to symmetry:

- (000) *is realizable if and only if* $\max\{x,y\} \leq u$, $\max\{x,z\} \leq v$ *and* $\max\{y,z\} \leq w$, *where*
  - *at most two of the inequalities can be equalities, and*
  - *if two are equalities, then either* x, y, z *are distinct and the edge (among* u, v, w*) that connects the shortest two of* x, y, z *attains equality, or* $\max\{x, y, z\}$ *is attained exactly twice, and the edge connecting those two longest does not attain equality.*
- (020) *is realizable if and only if* $v \leq u$, $y \leq z$, *and* $w + \max\{v, y\} \leq x$, *and if the last inequality is an equality then* $v = u$ *implies* $v < y < z$, *and* $y = z$ *implies* $y < v < u$.
- (111) *is realizable if and only if* $w < x$ *and*

$$(v + w < x \leq v + 3w \text{ and } v \leq u) \text{ or } (v + 3w < x \leq v + 4w \text{ and } v \leq u \leq 3v/2)$$
$$\text{or } (v + w = x \text{ and } v < u) \text{ or } (v + 4w < x \leq v + 5w \text{ and } v = u).$$

- (212) *is realizable if and only if* $w < x \leq 2w$.

Genus 3: Example



(212) is realizable if and only if $w < x \leq 2w$.

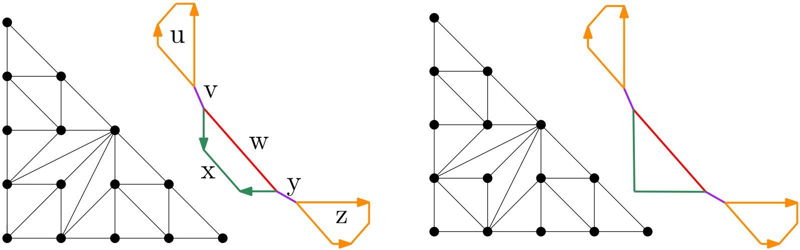


Figure: Triangulations that realize $w < x < 2w$ resp. $x = 2w$.

Genus 3: Probabilities

We define a probability measure on moduli space $\mathbb{M}_3$ as follows.

All five trivalent graphs G are equally likely. Non-trivalent graphs have probability 0. Each trivalent G gives an orthant $\mathbb{R}^6_{\geq 0}$, with measure induced from uniform distribution on the 5-simplex $\{x+y+z+u+v+w = 1\}$.

With this measure,

$$\frac{\mathrm{vol}(\mathbb{M}_3^{\mathrm{planar}})}{\mathrm{vol}(\mathbb{M}_3)} \quad = \quad \frac{31}{105}.$$

This was computed using our characterization in the Theorem:

Graph	(000)	(020)	(111)	(212)	(303)
Probability	4/15	8/15	12/35	1/3	0

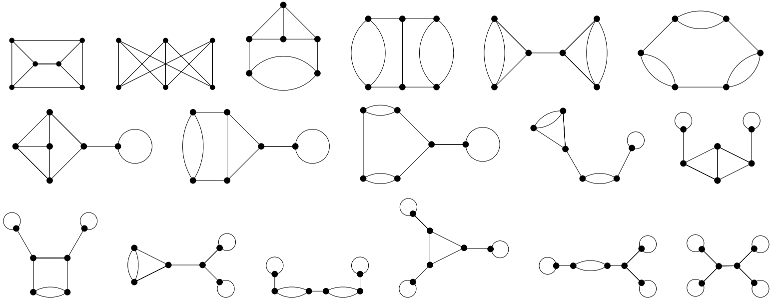
Corollary

Only 29.5% of all metric graphs of genus 3 come from quartics.

Genus 4

Theorem

Of the 17 trivalent graphs, precisely 13 are realizable by tropical plane curves. The moduli space $\mathbb{M}_4^{\text{planar}}$ is 9-dimensional, but it is not pure: there are maximal cones $\mathbb{M}_\Delta$ of dimensions 7, 8 and 9.

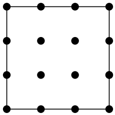


Corollary

Less than 0.5% of all metric graphs of genus 4 come from tropical plane curves. In fact, $\text{vol}(\mathbb{M}_4^{\text{planar}})/\text{vol}(\mathbb{M}_4) \approx 0.004641$.

Genus 4

In classical algebraic geometry, 100% of curves with $g = 4$ can be realized as plane curves with the square as the Newton polytope:



Reason: Canonical curves of genus 4*.*

In tropical algebraic geometry, that fraction is less than 0.5%.

Only five of the 13 realizable graphs contribute positive probability:

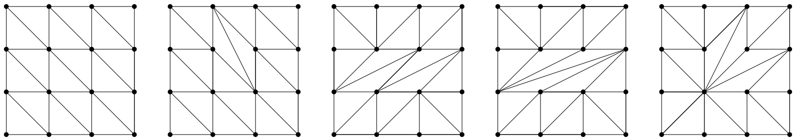
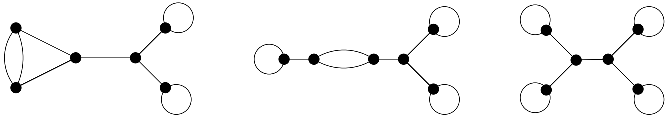


Figure: Triangulations Δ with $\dim(\mathbb{M}_{\Delta}) = 9$.

Sprawling

A connected, trivalent, leafless graph G is *sprawling* if there exists a vertex $v \in V(G)$ such that $G \backslash \{v\}$ consists of three distinct connected components.



Proposition

No sprawling graphs is the skeleton of a plane tropical curve.

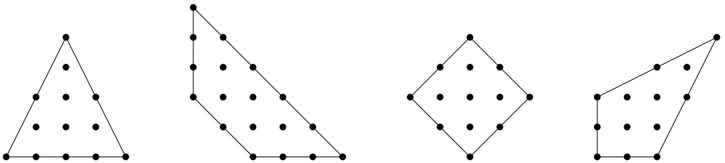
Philosophy: Experimental Mathematics with "Big Data" leads to new concepts and theoretical insights.

Genus Five

The moduli space of tropical plane curves of genus 5 is

$$\mathbb{M}_5^{\mathrm{planar}} \;=\; \mathbb{M}_{Q_1} \cup \mathbb{M}_{Q_2} \cup \mathbb{M}_{Q_3} \cup \mathbb{M}_{Q_4} \;\cup\; \mathbb{M}_{5,\mathrm{hyp}}^{\mathrm{planar}}$$

where Q_1, Q_2, Q_3, Q_4 are the polygons



Modulo their respective symmetries, the numbers of unimodular triangulations of these polygons are 508, 147908, 162 and 968.

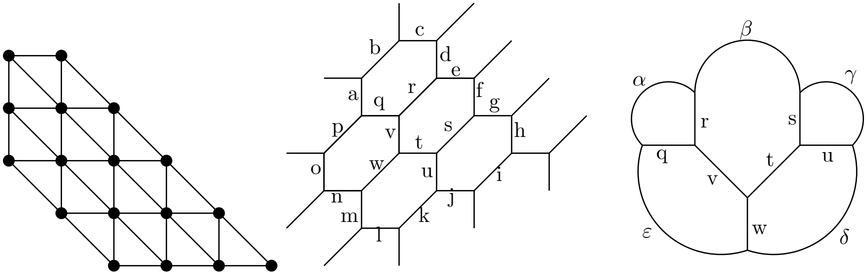
Theorem

Of the 71 *trivalent graphs, precisely* 38 *are realizable by tropical plane curves. There are obstructions other than sprawling. The stacky fan* $\mathbb{M}_5^{\mathrm{planar}}$ *has maximal cones* $\mathbb{M}_\Delta$ *of dimensions* 6 *to* 11*.*

Conclusion

Understanding how plane curves map into the moduli space of all curves is a classical topic in algebraic geometry.

In this lecture we studied this topic for tropical geometry, via the combinatorial link between tropical plane curves and metric graphs.



Have you found that linear relation among the 12 edge lengths?

And, yes, please buy our book....