Card-shuffling via convolutions of projections on combinatorial Hopf algebras

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Part I: The riffle-shuffle

- Cut the deck with symmetric binomial distribution;

$$\left.\begin{array}{l} i\left\{\begin{array}{l} 1\heartsuit \\ 2\diamondsuit \\ 3\heartsuit \end{array}\right. \\ \phantom{i\{}\ 4\spadesuit \\ \phantom{i\{}\ 5\spadesuit \end{array}\right\} n$$

$$\text{Prob} = 2^{-n}\binom{n}{i}$$

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$2\diamondsuit$	$5\spadesuit$
$3\heartsuit$	

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- Drop one-by-one the bottommost card, from a pile chosen with probability proportional to current pile size.

$1\heartsuit$ $4\spadesuit$

$2\diamondsuit$ $5\spadesuit$ Prob $= \frac{2}{5}$

Prob $= \frac{3}{5}$ $3\heartsuit$

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$1\heartsuit$ $4\spadesuit$ Prob $= \frac{1}{4}$

$2\diamondsuit$

Prob $= \frac{3}{4}$ $3\heartsuit$

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Prob $= \frac{1}{2}$ $1\heartsuit$ $4\spadesuit$ Prob $= \frac{1}{2}$

$2\diamondsuit$

$3\heartsuit$

$5\spadesuit$

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Prob $= \frac{1}{1}$ $1\heartsuit$

$4\spadesuit$

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Equivalently, all interleavings are equally likely.

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Bayer-Diaconis (1992):
Randomising n distinct cards needs $\frac{3}{2}\log n$ shuffles.

A new tool: the shuffle (Hopf) algebra $\mathcal{S}$

- graded: $\mathcal{S} = \bigoplus \mathcal{S}_n$
- basis of $\mathcal{S}_n$ is $\mathcal{B}_n$:={words of length n}={decks of n cards}
  (possibly with repeated letters / cards)

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- product $m : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ is sum of all interleavings

  $m([15] \otimes [5]) = [155] + [155] + [515]$

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  $$m([15] \otimes [5]) = \qquad 2[155] + [515]$$

- coproduct $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ is sum of all deconcatenations

  $$\Delta([155]) = \epsilon \otimes [155] + [1] \otimes [55] + [15] \otimes [5] + [155] \otimes \epsilon$$

  ↑ empty deck = unit of $\mathcal{S}$

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Relation with riffle-shuffling:

Prob$(x \to y)$ = coefficient of y in $\frac{1}{2^n} m \circ \Delta(x)$ for $x, y \in \mathcal{B}_n$.

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Relation with riffle-shuffling:

Prob$(x \to y)$ = coefficient of y in $\frac{1}{2^n} m \circ \Delta(x)$ for $x, y \in \mathcal{B}_n$.

$$\tfrac{1}{8} m \circ \Delta([155]) = \frac{1}{8} \begin{pmatrix} [155] + ([155] + [515] + [551]) \\ +(2[155] + [515]) + [155] \end{pmatrix}$$

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$$= \frac{5}{8}[155] + \frac{2}{8}[515] + \frac{1}{8}[551]$$

Consequences

$$\text{Prob}(x \to y) = \text{coefficient of } y \text{ in } \frac{1}{2^n} m \circ \Delta(x) \text{ for } x, y \in \mathcal{B}_n$$

Theorem (w/ Diaconis, Ram, 2014): Algorithm for a basis of eigenvectors of $m \circ \Delta$ on shuffle algebra, from Hopf algebraic structure theorems.

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Corollary (and folklore): Stationary distribution is uniform.

Corollary (and folklore): Start with n distinct cards in ascending order. After t riffle-shuffles:

$$\text{Expect } \{\text{number of descents}\} = \left(1 - \left(\frac{1}{2}\right)^t\right) \frac{n-1}{2}.$$

↑ high card on low card

Other shuffling schemes

Prob$(x \to y)$ = coefficient of y in $\mathbf{T}(x)$ for $x, y \in \mathcal{B}_n$.

Riffle-shuffle $\mathbf{T} = \frac{1}{2^n} m \circ \Delta$

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Riffle-shuffle $\qquad \mathbf{T} = \frac{1}{2^n} m \circ \Delta$

Top-to-random $\qquad \mathbf{T} = \frac{1}{n} m \circ \Delta_{1,n-1}$

Project the coproduct to $\mathcal{S}_1 \otimes \mathcal{S}_{n-1}$

$$\Delta([155]) = \emptyset \otimes [155] + [1] \otimes [55] + [15] \otimes [5] + [155] \otimes \emptyset$$

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Biased cut riffle	$\mathbf{T} = \sum q^i (1-q)^{n-i} m \circ \Delta_{i,n-i}$

$$(1-q)^3 \qquad q(1-q)^2 \qquad q^2(1-q) \qquad q^3$$

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cut-and-interleave

Diaconis, Fill, Pitman (1992)

descent operators

Patras (1994)

Consequences

Prob$(x \to y)$ = coefficient of y in $\mathbf{T}(x)$ for $x, y \in \mathcal{B}_n$.

Theorem (2015): For many significant $\mathbf{T}$ (top-to-random, top-or-bottom-to-random, etc.), we can algorithmically compute an eigenbasis.

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Theorem (2015): For many significant $\mathbf{T}$ (top-to-random, top-or-bottom-to-random, etc.), we can algorithmically compute an eigenbasis.

Corollary: Stationary distribution is always uniform.

Corollary: Start with n distinct cards in ascending order. After t top-to-random shuffles:

$$\text{Prob}\,\{\text{descent at bottom}\} = \left(1 - \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}.$$

Part II: Break-and-recombine other combinatorial objects

On other combinatorial Hopf algebras, define Markov chain by:

$\text{Prob}(x \to y) := $ coefficient of y in $\mathbf{T}(x)$ for $x, y \in \mathcal{B}_n$

$\mathbf{T} =$ descent operator $\sim$ style of shuffle

$\mathcal{B}_n =$ basis $\sim$ combinatorial object

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$\mathcal{B}_n$ = basis $\sim$ combinatorial object

shuffle algebra	$\longrightarrow$ card-shuffling
Connes-Kreimer trees	$\longrightarrow$ tree pruning
graph Hopf algebra	$\longrightarrow$ edge removal
symmetric functions, schur basis	$\longrightarrow$ a chain on partitions

Example: top-to-random on partitions

Prob$(\lambda \to \mu)$: $=$ coefficient of s_μ in $\frac{1}{n} m \circ \Delta_{1,n-1}(s_\lambda)$.

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$$\Delta_{1,n-1}(s_\lambda) = \sum_{\nu = \lambda \backslash \square} s_{(1)} \otimes s_\nu; \quad s_{(1)} s_\nu = \sum_{\mu = \nu \cup \square} s_\mu$$

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$$\frac{1}{3} m \circ \Delta_{1,2} \left(\square\square\square \right) = \frac{1}{3} m \left(\square \otimes \square\square \right)$$

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$$\frac{1}{3} m \circ \Delta_{1,2}\left(\yng(3)\right) = \frac{1}{3} m \left(\yng(1) \otimes \yng(2)\right) = \frac{1}{3}\,\yng(3) + \frac{1}{3}\,\yng(2,1)$$

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		μ		
		(3)	$(2,1)$	$(1,1,1)$
	(3)	$1/3$	$1/3$	
λ	$(2,1)$	$1/3$	$2/3$	$1/3$
	$(1,1,1)$		$1/3$	$1/3$

Example: top-to-random on partitions

Prob$(\lambda \to \mu)$: $=$ coefficient of ~~s_μ~~ in $\frac{1}{n} m \circ \Delta_{1,n-1}$(~~$s_\lambda$~~).

Divide s_λ by number of standard tableaux of shape λ

$$\frac{s_\mu}{\dim \mu} \qquad \frac{s_\lambda}{\dim \lambda}$$

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To make coefficients sum to 1, use "the Doob transform"

Example: top-to-random on partitions

Prob($\lambda \to \mu$): $=$ coefficient of ~~s_μ~~ $\frac{s_\mu}{\dim \mu}$ in $\frac{1}{n} m \circ \Delta_{1,n-1}($~~$s_\lambda$~~ $\frac{s_\lambda}{\dim \lambda}$$)$.

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$$\frac{1}{3} m \circ \Delta_{1,2}\left(\square\square\square\right) = \frac{1}{3} m \left(\square \otimes \square\square\right) = \frac{1}{3}\square\square\square + \frac{2}{3}\left(\frac{1}{2}\begin{array}{l}\square\square\\ \square\end{array}\right)$$

		μ	
	(3)	$(2,1)$	$(1,1,1)$
λ (3)	$1/3$	$2/3$	
λ $(2,1)$	$1/6$	$2/3$	$1/6$
λ $(1,1,1)$		$2/3$	$1/3$

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General theorems

$$\text{Prob}(x \to y) := \text{coefficient of } \frac{y}{\eta(y)} \text{ in } \mathbf{T}\left(\frac{x}{\eta(x)}\right) \text{ for } x, y \in \mathcal{B}_n$$

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All eigenvalues and multiplicities

$\dfrac{j}{n}$, #partitions with j parts of size 1

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Explicit expression for all stationary distributions (independent of $\mathbf{T}$)

Plancherel measure $\dfrac{(\dim \lambda)^2}{n!}$

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Quotient / sub Hopf algebras give lumpings (independent of $\mathbf{T}$)

RSK shape of top-to-random-with-standardisation
$=$ top-to-random on partitions, because
$\Lambda \twoheadleftarrow \mathbf{FSym} \hookrightarrow \mathbf{FQSym}$

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Please tell me your favourite Hopf algebras and non-negative linear maps

Thank you!

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shuffle

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