Pieri rule for the affine flag variety

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Overview

Classical Schubert calculus

Schubert calculus is a branch of enumerative algebraic geometry concerned with problems of the form:

How many lines L intersect four fixed lines $L_1, \ldots, L_4$ in complex (projective) 3-space?

The question is reduced to finding structure constants of the cohomology $H^*(Gr(m,n))$ of the Grassmannian parameterizing m-dimensional subspaces in $\mathbb{C}^n$.

Classical Schubert calculus

The Grassmannian $Gr(m, n)$ has distinguished Schubert basis Ω_λ indexed by partitions λ contained in $m \times (n - m)$.

Theorem

Let Λ be the ring of symmetric functions in $x_1, x_2, \cdots$. Then there is a surjective ring homomorphism $p : \Lambda \to H^*(Gr(m, n))$ defined by

$$p(s_\lambda) = \begin{cases} [\Omega_\lambda] & \text{if } \lambda \subset m \times (n-m) \\ 0 & \text{otherwise.} \end{cases}$$

where s_λ is the Schur function indexed by λ.

Classical Schubert calculus

For $m < n$ and λ, μ, ν are partitions contained in $m \times (n-m)$, we have

$$[\Omega_\mu][\Omega_\nu] = \sum_\lambda c^\lambda_{\mu,\nu}[\Omega_\lambda]$$

where $c^\lambda_{\mu,\nu}$ is called the structure constants of $H^*(Gr(m,n))$.

From the previous theorem, the structure constants are the same as the Littlewood-Richardson coefficients:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

where λ, μ, ν are partitions.

Pieri rule

Pieri rule is one of the useful formulas for computing Littlewood-Richardson coefficients.

Theorem (Pieri rule)

For a positive integer m, let h_m be the homogeneous polynomial of degree m. Then we have

$$h_m s_\lambda = \sum s_\mu$$

where μ are obtained by adding a horizontal strip of size m from λ.

Affine Schubert calculus

Let $\Lambda_{(k)}$ denote the subalgebra generated by $h_1, h_2, \ldots, h_k$, and let $\Lambda^{(k)} = \Lambda/I_k$ denote the quotient of Λ by the ideal I_k generated by m_λ with $\lambda_1 > k$.

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Theorem (Lam, 2008)

Let **Gr** *denote the affine Grassmannian associated with* $SL(n, \mathbb{C})$. *There are isomorphism of Hoph-algebras*

$$H_*(\mathbf{Gr}) \cong \Lambda_{(k)}$$
$$H^*(\mathbf{Gr}) \cong \Lambda^{(k)},$$

and image of the Schubert basis in $H_*(\mathbf{Gr})$ *(resp.* $H^*(\mathbf{Gr})$*) via the isomorphism is a k-Schur fuction (resp. an affine Stanley symmetric function).*

Comparison

Let Fl denote the affine flag variety associated with $SL(n, \mathbb{C})$, which is the affine type A version of the flag variety.

Geometry	Index set	Polynomial representative
$H^*(Gr(m, n))$	partitions $\subset m \times (n - m)$	Schur functions
H^*(finite flag)	symmetric group	Schubert polynomials
$H_*(\mathbf{Gr})$	0-Grassmannian elements	k-Schur functions
$H^*(\mathbf{Gr})$	0-Grassmannian elements	affine Stanley symmetric functions
$H_*(Fl)$	affine symmetric group	?
$H^*(Fl)$	affine symmetric group	?

Comparison

Classical case		Affine case
partitions	$\longrightarrow$	0-Grassmannian elements
symmetric group	$\longrightarrow$	affine symmetric group
content	$\longrightarrow$	marking
horizontal strips	$\longrightarrow$	strong strips
semistandard Young tableaux	$\longrightarrow$	strong tableaux
Schur functions	$\longrightarrow$	Strong schur functions

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Remark

k-Schur functions are the special cases of the strong Schur functions.

Affine symmetric group

Let $\widetilde{S_n}$ denote the affine symmetric group with simple generators $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\ s_i s_j &= s_j s_i && \text{if } |i-j| \geq 2 \end{aligned}$$

where indices are taken modulo n.

An element of the affine symmetric group may be written as a word in the generators s_i. A *reduced word* of the element is a word of minimal length. The *length* of w, denoted $\ell(w)$, is the number of generators in any reduced word of w.

Affine symmetric group

The *Bruhat order*, also called *strong order*, on $\widetilde{S_n}$ is a partial order where $u < w$ if there is a reduced word for u that is a subword of a reduced word for w. If $u < w$ and $\ell(u) = \ell(w) - 1$, we write $u \lessdot w$. It is well-known that $u \lessdot w$ if and only if there exists a transposition t such that $w = ut$ and $\ell(w) = \ell(u) - 1$.

Example

$s_1 s_0 \lessdot s_2 s_1 s_0 = s_1 s_0 \cdot t_{03}$.

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The subgroup of $\widetilde{S_n}$ generated by $\{s_1, \cdots, s_{n-1}\}$ is naturally isomorphic to the symmetric group S_n. The 0-Grassmannian elements are minimal length coset representatives of $\widetilde{S_n}/S_n$. An element w is 0-Grassmannian if and only if all reduced words of w end with s_0.

Strong strips

A *marked strong cover* $C = \left(u \xrightarrow{a} v\right)$ consists of $u, v \in \widetilde{S_n}$ and an integer a such that $u = vt_{ij}, v \lessdot u, a = u(j) = v(i)$ where $i \leq 0 < j$. Here, a is called the marking of C.

Definition

A *strong strip* S of length i from u to v, denoted by $u \longrightarrow v$, is a path

$$u \xrightarrow{a_1} u_1 \xrightarrow{a_2} \cdots \xrightarrow{a_i} u_i = v$$

where $a_1 > a_2 > \ldots > a_i$.

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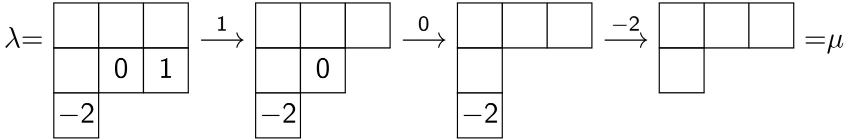
A *content* of a box at (i,j) in a Young diagram is $j-i$. It is well-known that a skew partition λ/μ is a horizontal strip if and only if there exists a chain of the (order-reversed) Bruhat covers from λ to μ with decreasing contents.

Example



Note the skew shape $(3,3,1)/(3,1)$ is a horizontal strip and the contents removed from the above sequence are decreasing.

Pieri rule for the affine Grassmannian

For $i > 0$, let ρ_i be the Pieri factor defined by $s_{i-1}s_{i-2}\dots s_1s_0$.

Theorem (Lam, Lapointe, Morse, Shimozono, 2010)

Let ξ^w be the Schubert class in the cohomology of the affine Grassmannian. Then for 0-Grassmannian element w, we have

$$\xi^{\rho_i}\xi^w = \sum \xi^u$$

where the sum runs over all strong strips of length i from u to w.

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Recall that the Schubert class ξ^w can be identified with the affine Stanley symmetric function $\tilde{F}_w$.
The affine Stanley symmetric fuction $\tilde{F}_{\rho_i}$ for ρ_i is the image of h_i in $\Lambda^{(k)}$.

Pieri rule for the affine flag variety

Conjecture (Lam, Lapointe, Morse, Shimozono, 2010)

Let ζ^w be the Schubert class in the cohomology of the affine flag variety. Then for $w \in \widetilde{S_n}$, we have

$$\zeta^{\rho_i}\zeta^w = \sum \zeta^u$$

where the sum runs over all strong strips of length i from u to w.

One of obstacles: absence of polynomial representative.

Pieri rule for the affine flag variety

Theorem (L., 2014)

The conjectured Pieri rule for the affine flag variety holds.

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The conjectured Pieri rule for the affine flag variety holds.

For $v, w, u \in \widetilde{S_n}$, let $p^v_{w,u}$ be the structure constants of $H^*(Fl)$.

Theorem (Reformulation)

For $w, u \in \widetilde{S_n}$ with $i = \ell(w) - \ell(u) \in \mathbb{N}$, $p^w_{\rho_i,u}$ counts the number of strong strips of length i from w to u.

Ingredients of the proof:

1. Affine nilCoxeter algebra, providing a way to compute the structure constants $p^v_{w,u}$.
2. Pieri operators defined by Berg, Saliola, Serrano.
3. the cap operator for ρ_i using the Kostant and Kumar's work.

Both operators are defined on the affine nilCoxeter algebra.

Affine nilCoxeter algebra

The affine nilCoxeter algebra $\mathbb{A}$ is an algebra with generators $A_{s_0}, A_{s_1}, \ldots, A_{s_{n-1}}$ with certain relations, and is isomorphic to

$$\bigoplus_{w \in \widetilde{S_n}} \mathbb{Z} A_w$$

as a $\mathbb{Z}$-module. We identify $\mathbb{A}$ with the homology of the affine flag variety.

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as a $\mathbb{Z}$-module. We identify $\mathbb{A}$ with the homology of the affine flag variety. There is a coproduct structure on the affine nilCoxeter algebra $\mathbb{A}$.

Theorem (Kostant, Kumar, 1986)

For $v \in \widetilde{S_n}$,

$$\Delta(A_v) = \sum_{w,u} p^v_{w,u} A_w \otimes A_u.$$

where the sum runs over all $w, u \in \widetilde{S_n}$ satisfying $\ell(v) = \ell(w) + \ell(u)$.

Pieri operators and cap operators

For $w \in \widetilde{S_n}$, let us define *the cap operator* D_w on $\mathbb{A}$ by

$$D_w(A_v) := \sum_{\substack{u \\ \ell(u)=\ell(w)-\ell(v)}} p^v_{w,u} A_u.$$

Geometrically, D_w is the cap product on the homology of the affine flag variety. Indeed, $D_w(A_v)$ can be identified with $\zeta^w \cap \zeta_v$. Let D_i denote D_{ρ_i}.

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On the other hand, Berg, Saliola, Serrano studied the Pieri operator D'_i by

$$D'_i(A_w) = \sum_{w \to u} A_u$$

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Theorem (L.)

$D_i = D'_i$.

Identities related to cap operators

The identification $D_i = D'_i$ follows from the fact that both operators satisfy identities that uniquiely determine the operators.

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There is a commutative subalgebra $\mathbb{B}$ of $\mathbb{A}$, called the affine Fomin-Stanley algebra, which is isomorphic to $\Lambda_{(k)}$.

Theorem

The cap operator D_i stabilizes $\mathbb{B}$. As an action on $\mathbb{B}$, we have $D_i = h_i^{\perp}$ where $h_i^{\perp}$ is the adjoint action of the multiplication by h_i.
(For $g \in \Lambda_{(k)}$ and $f, h \in \Lambda^{(k)}$, $\langle f^{\perp}(g), h\rangle = \langle g, fh\rangle$.)

Identities related to cap operators

Theorem

For $w \in \widetilde{S_n}$, the cap operator D_w stabilizes $\mathbb{B}$. As an action on $\mathbb{B}$, we have $D_w = \tilde{F}_w^{\perp}$.

In fact, some properties of D_w can be generalized to other affine types.

Lemma

For $f, g \in \mathbb{B}$ and $w \in \widetilde{S_n}$, we have

$$D_w(fg) = \sum_{\substack{u,v \\ \ell(w)=\ell(u)+\ell(v)}} D_u(f) D_v(g).$$

Question

Can we construct certain combinatorics similar to strong/weak tableaux for other types?

Related work:

- Pon introduced affine Stanley symmetric functions for the special orthogonal groups and Pieri factor generalizing ρ_i.
- Lam and Shimozono defined the analogue of the marked strong cover and weak cover for an arbitrary Kac-Moody algebra.

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Thank you!