Pieri rule for the affine flag variety

Seung Jin Lee

KIAS

FPSAC, July 9, 2015
Overview

1. Introduction
   - Classical Schubert calculus
   - Affine Schubert calculus

2. Combinatorics on the affine symmetric group

3. Pieri rule for the affine Grassmannian

4. Pieri rule for the affine flag variety

5. Operators on the affine nilCoxeter algebra

6. Concluding remark
Schubert calculus is a branch of enumerative algebraic geometry concerned with problems of the form:

How many lines $L$ intersect four fixed lines $L_1, \ldots, L_4$ in complex (projective) 3-space?

The question is reduced to finding structure constants of the cohomology $H^*(Gr(m, n))$ of the Grassmannian parameterizing $m$-dimensional subspaces in $\mathbb{C}^n$. 
The Grassmannian $Gr(m, n)$ has distinguished Schubert basis $\Omega_\lambda$ indexed by partitions $\lambda$ contained in $m \times (n - m)$.

**Theorem**

Let $\Lambda$ be the ring of symmetric functions in $x_1, x_2, \cdots$. Then there is a surjective ring homomorphism $p : \Lambda \to H^*(Gr(m, n))$ defined by

$$p(s_\lambda) = \begin{cases} [\Omega_\lambda] & \text{if } \lambda \subset m \times (n - m) \\ 0 & \text{otherwise.} \end{cases}$$

where $s_\lambda$ is the Schur function indexed by $\lambda$. 
Classical Schubert calculus

For $m < n$ and $\lambda, \mu, \nu$ are partitions contained in $m \times (n - m)$, we have

$$[\Omega_\mu][\Omega_\nu] = \sum_{\lambda} c_{\mu, \nu}^\lambda [\Omega_\lambda]$$

where $c_{\mu, \nu}^\lambda$ is called the structure constants of $H^*(Gr(m, n))$.

From the previous theorem, the structure constants are the same as the Littlewood-Richardson coefficients:

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu, \nu}^\lambda s_\lambda.$$

where $\lambda, \mu, \nu$ are partitions.
Pieri rule is one of the useful formulas for computing Littlewood-Richardson coefficients.

Theorem (Pieri rule)

For a positive integer $m$, let $h_m$ be the homogeneous polynomial of degree $m$. Then we have

$$h_m s_\lambda = \sum s_\mu$$

where $\mu$ are obtained by adding a horizontal strip of size $m$ from $\lambda$. 

Affine Schubert calculus

Let $\Lambda^{(k)}$ denote the subalgebra generated by $h_1, h_2, \ldots, h_k$, and let $\Lambda^{(k)} = \Lambda/I_k$ denote the quotient of $\Lambda$ by the ideal $I_k$ generated by $m_{\lambda}$ with $\lambda_1 > k$. 

Theorem (Lam, 2008)

Let $Gr$ denote the affine Grassmannian associated with $SL(n, \mathbb{C})$. There are isomorphisms of Hopf-algebras $H^*(Gr) \cong \Lambda^{(k)}$. The image of the Schubert basis in $H^*(Gr)$ (resp. $H^*(Gr)$) via the isomorphism is a $k$-Schur function (resp. an affine Stanley symmetric function).
Affine Schubert calculus

Let $\Lambda^{(k)}$ denote the subalgebra generated by $h_1, h_2, \ldots, h_k$, and let $\Lambda^{(k)} = \Lambda/I_k$ denote the quotient of $\Lambda$ by the ideal $I_k$ generated by $m_\lambda$ with $\lambda_1 > k$.

Theorem (Lam, 2008)

Let $\text{Gr}$ denote the affine Grassmannian associated with $SL(n, \mathbb{C})$. There are isomorphism of Hopf-algebras

$$H_*(\text{Gr}) \cong \Lambda^{(k)}$$

$$H^*(\text{Gr}) \cong \Lambda^{(k)},$$

and image of the Schubert basis in $H_*(\text{Gr})$ (resp. $H^*(\text{Gr})$) via the isomorphism is a $k$-Schur function (resp. an affine Stanley symmetric function).
Let $Fl$ denote the affine flag variety associated with $SL(n, \mathbb{C})$, which is the affine type A version of the flag variety.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Index set</th>
<th>Polynomial representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^*(Gr(m, n))$</td>
<td>partitions $\subset m \times (n - m)$</td>
<td>Schur functions</td>
</tr>
<tr>
<td>$H^*(\text{finite flag})$</td>
<td>symmetric group</td>
<td>Schubert polynomials</td>
</tr>
<tr>
<td>$H_*(Gr)$</td>
<td>0-Grassmannian elements</td>
<td>$k$-Schur functions</td>
</tr>
<tr>
<td>$H^*(Gr)$</td>
<td>0-Grassmannian elements</td>
<td>affine Stanley symmetric functions</td>
</tr>
<tr>
<td>$H_*(Fl)$</td>
<td>affine symmetric group</td>
<td></td>
</tr>
<tr>
<td>$H^*(Fl)$</td>
<td>affine symmetric group</td>
<td>?</td>
</tr>
</tbody>
</table>
## Comparison

<table>
<thead>
<tr>
<th>Classical case</th>
<th>Affine case</th>
</tr>
</thead>
<tbody>
<tr>
<td>partitions</td>
<td>0-Grassmannian elements</td>
</tr>
<tr>
<td>symmetric group</td>
<td>affine symmetric group</td>
</tr>
<tr>
<td>content</td>
<td>marking</td>
</tr>
<tr>
<td>horizontal strips</td>
<td>strong strips</td>
</tr>
<tr>
<td>semistandard Young tableaux</td>
<td>strong tableaux</td>
</tr>
<tr>
<td>Schur functions</td>
<td>Strong schur functions</td>
</tr>
</tbody>
</table>
## Comparison

<table>
<thead>
<tr>
<th>Classical case</th>
<th>Affine case</th>
</tr>
</thead>
<tbody>
<tr>
<td>partitions</td>
<td>0-Grassmannian elements</td>
</tr>
<tr>
<td>symmetric group</td>
<td>affine symmetric group</td>
</tr>
<tr>
<td>content</td>
<td>marking</td>
</tr>
<tr>
<td>horizontal strips</td>
<td>strong strips</td>
</tr>
<tr>
<td>semistandard Young tableaux</td>
<td>strong tableaux</td>
</tr>
<tr>
<td>Schur functions</td>
<td>Strong schur functions</td>
</tr>
</tbody>
</table>

### Remark

*k-Schur functions are the special cases of the strong Schur functions.*
Let $\widetilde{S}_n$ denote the affine symmetric group with simple generators $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

\[
\begin{align*}
  s_i^2 &= 1 & \text{for all } i \\
  s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} & \text{for all } i \\
  s_is_j &= s_js_i & \text{if } |i-j| \geq 2
\end{align*}
\]

where indices are taken modulo $n$.

An element of the affine symmetric group may be written as a word in the generators $s_i$. A \textit{reduced word} of the element is a word of minimal length. The \textit{length} of $w$, denoted $\ell(w)$, is the number of generators in any reduced word of $w$. 
The *Bruhat order*, also called *strong order*, on $\tilde{S}_n$ is a partial order where $u < w$ if there is a reduced word for $u$ that is a subword of a reduced word for $w$. If $u < w$ and $\ell(u) = \ell(w) - 1$, we write $u \lessdot w$. It is well-known that $u \lessdot w$ if and only if there exists a transposition $t$ such that $w = ut$ and $\ell(w) = \ell(u) - 1$.

**Example**

$s_1s_0 \lessdot s_2s_1s_0 = s_1s_0 \cdot t_{03}$.
The *Bruhat order*, also called *strong order*, on $\widetilde{S}_n$ is a partial order where $u < w$ if there is a reduced word for $u$ that is a subword of a reduced word for $w$. If $u < w$ and $\ell(u) = \ell(w) - 1$, we write $u \preceq w$. It is well-known that $u \preceq w$ if and only if there exists a transposition $t$ such that $w = ut$ and $\ell(w) = \ell(u) - 1$.

**Example**

$s_1 s_0 \preceq s_2 s_1 s_0 = s_1 s_0 \cdot t_{03}$.

The subgroup of $\widetilde{S}_n$ generated by $\{s_1, \cdots, s_{n-1}\}$ is naturally isomorphic to the symmetric group $S_n$. The 0-Grassmannian elements are minimal length coset representatives of $\widetilde{S}_n/S_n$. An element $w$ is 0-Grassmannian if and only if all reduced words of $w$ end with $s_0$. 
Strong strips

A marked strong cover $C = (u \xrightarrow{a} v)$ consists of $u, v \in \widetilde{S}_n$ and an integer $a$ such that $u = v t_{ij}, v \preceq u, a = u(j) = v(i)$ where $i \leq 0 < j$. Here, $a$ is called the marking of $C$.

Definition

A strong strip $S$ of length $i$ from $u$ to $v$, denoted by $u \rightarrow v$, is a path

\[ u \xrightarrow{a_1} u_1 \xrightarrow{a_2} \cdots \xrightarrow{a_i} u_i = v \]

where $a_1 > a_2 > \ldots > a_i$. 
<table>
<thead>
<tr>
<th>Classical case</th>
<th>Affine case</th>
</tr>
</thead>
<tbody>
<tr>
<td>content</td>
<td>marking</td>
</tr>
<tr>
<td>horizontal strips</td>
<td>strong strips</td>
</tr>
</tbody>
</table>
A content of a box at \((i, j)\) in a Young diagram is \(j - i\). It is well-known that a skew partition \(\lambda/\mu\) is a horizontal strip if and only if there exists a chain of the (order-reversed) Bruhat covers from \(\lambda\) to \(\mu\) with decreasing contents.

Example

\[
\begin{array}{ccccccc}
0 & 1 \\
-2 & & & & & \\
\end{array} \quad \xrightarrow{1} \quad \begin{array}{ccccccc}
0 & & & & & \\
-2 & & & & & \\
\end{array} \quad \xrightarrow{0} \quad \begin{array}{ccccccc}
& & & & & \\
& & & & & \\
\end{array} \quad \xrightarrow{-2} \quad \begin{array}{ccccccc}
& & & & & \\
& & & & & \\
\end{array} = \mu
\]

Note the skew shape \((3, 3, 1)/(3, 1)\) is a horizontal strip and the contents removed from the above sequence are decreasing.
For $i > 0$, let $\rho_i$ be the Pieri factor defined by $s_{i-1}s_{i-2} \ldots s_1s_0$.

Theorem (Lam, Lapointe, Morse, Shimozono, 2010)

Let $\xi^w$ be the Schubert class in the cohomology of the affine Grassmannian. Then for 0-Grassmannian element $w$, we have

$$\xi^{\rho_i} \xi^w = \sum \xi^u$$

where the sum runs over all strong strips of length $i$ from $u$ to $w$. 
For $i > 0$, let $\rho_i$ be the Pieri factor defined by $s_i s_{i-2} \ldots s_1 s_0$.

**Theorem (Lam, Lapointe, Morse, Shimozono, 2010)**

Let $\xi^w$ be the Schubert class in the cohomology of the affine Grassmannian. Then for $0$-Grassmannian element $w$, we have

$$\xi^{\rho_i} \xi^w = \sum \xi^u$$

where the sum runs over all strong strips of length $i$ from $u$ to $w$.

Recall that the Schubert class $\xi^w$ can be identified with the affine Stanley symmetric function $\tilde{F}_w$.

The affine Stanley symmetric function $\tilde{F}_{\rho_i}$ for $\rho_i$ is the image of $h_i$ in $\Lambda^{(k)}$. 
Conjecture (Lam, Lapointe, Morse, Shimozono, 2010)

Let $\zeta^w$ be the Schubert class in the cohomology of the affine flag variety. Then for $w \in \widetilde{S}_n$, we have

$$\zeta^\rho_i \zeta^w = \sum \zeta^u$$

where the sum runs over all strong strips of length $i$ from $u$ to $w$.

One of obstacles: absence of polynomial representative.
Theorem (L., 2014)

The conjectured Pieri rule for the affine flag variety holds.
Theorem (L., 2014)

The conjectured Pieri rule for the affine flag variety holds.

For \(v, w, u \in \widetilde{S}_n\), let \(p_{w,u}^v\) be the structure constants of \(H^*(Fl)\).

Theorem (Reformulation)

For \(w, u \in \widetilde{S}_n\) with \(i = \ell(w) - \ell(u) \in \mathbb{N}\), \(p_{\rho_i,u}^w\) counts the number of strong strips of length \(i\) from \(w\) to \(u\).
Ingredients of the proof:

1. Affine nilCoxeter algebra, providing a way to compute the structure constants $p^v_{w,u}$.
2. Pieri operators defined by Berg, Saliola, Serrano.
3. The cap operator for $\rho_i$ using the Kostant and Kumar’s work.

Both operators are defined on the affine nilCoxeter algebra.
The affine nilCoxeter algebra $\mathbb{A}$ is an algebra with generators $A_{s_0}, A_{s_1}, \ldots, A_{s_{n-1}}$ with certain relations, and is isomorphic to

$$\bigoplus_{w \in \tilde{S}_n} \mathbb{Z}A_w$$

as a $\mathbb{Z}$-module. We identify $\mathbb{A}$ with the homology of the affine flag variety.
The affine nilCoxeter algebra $\mathbb{A}$ is an algebra with generators $A_{s_0}, A_{s_1}, \ldots, A_{s_{n-1}}$ with certain relations, and is isomorphic to

$$\bigoplus_{w \in \tilde{S}_n} \mathbb{Z}A_w$$

as a $\mathbb{Z}$-module. We identify $\mathbb{A}$ with the homology of the affine flag variety. There is a coproduct structure on the affine nilCoxeter algebra $\mathbb{A}$.

**Theorem (Kostant, Kumar, 1986)**

For $v \in \tilde{S}_n$,

$$\Delta(A_v) = \sum_{w,u} p^v_{w,u} A_w \otimes A_u.$$

where the sum runs over all $w, u \in \tilde{S}_n$ satisfying $\ell(v) = \ell(w) + \ell(u)$. 

For $w \in \widetilde{S}_n$, let us define the cap operator $D_w$ on $\mathbb{A}$ by

$$D_w(A_v) := \sum_{u \text{ s.t. } \ell(u) = \ell(w) - \ell(v)} p_{w,u}^v A_u.$$  

Geometrically, $D_w$ is the cap product on the homology of the affine flag variety. Indeed, $D_w(A_v)$ can be identified with $\zeta^w \cap \zeta^v$. Let $D_i$ denote $D_{\rho_i}$. 


For \( w \in \widetilde{S}_n \), let us define the cap operator \( D_w \) on \( A \) by

\[
D_w(A_v) := \sum_{u \in U} p_{w,u}^v A_u.
\]

Geometrically, \( D_w \) is the cap product on the homology of the affine flag variety. Indeed, \( D_w(A_v) \) can be identified with \( \zeta_w \cap \zeta_v \). Let \( D_i \) denote \( D_{\rho_i} \).

On the other hand, Berg, Saliola, Serrano studied the Pieri operator \( D_i' \) by

\[
D_i'(A_w) = \sum_{w \rightarrow u} A_u
\]

where \( w \rightarrow u \) is the strong strip of length \( i \).
Pieri operators and cap operators

For $w \in \tilde{S}_n$, let us define the cap operator $D_w$ on $\mathbb{A}$ by

$$D_w(A_v) := \sum_{u} p_{w,u}^v A_u.$$ 

Geometrically, $D_w$ is the cap product on the homology of the affine flag variety. Indeed, $D_w(A_v)$ can be identified with $\zeta_w \cap \zeta_v$. Let $D_i$ denote $D_{\rho_i}$.

On the other hand, Berg, Saliola, Serrano studied the Pieri operator $D'_i$ by

$$D'_i(A_w) = \sum_{w \to u} A_u$$

where $w \to u$ is the strong strip of length $i$.

Theorem (L.)

$$D_i = D'_i.$$
The identification $D_i = D'_i$ follows from the fact that both operators satisfy identities that uniquely determine the operators.
The identification $D_i = D'_i$ follows from the fact that both operators satisfy identities that uniquely determine the operators.

There is a commutative subalgebra $\mathbb{B}$ of $\mathbb{A}$, called the affine Fomin-Stanley algebra, which is isomorphic to $\Lambda^{(k)}$.

**Theorem**

The cap operator $D_i$ stabilizes $\mathbb{B}$. As an action on $\mathbb{B}$, we have $D_i = h_i^\perp$ where $h_i^\perp$ is the adjoint action of the multiplication by $h_i$.

(For $g \in \Lambda^{(k)}$ and $f, h \in \Lambda^{(k)}$, $\langle f^\perp(g), h \rangle = \langle g, fh \rangle$.)
Theorem

For \( w \in \widetilde{S}_n \), the cap operator \( D_w \) stabilizes \( \mathbb{B} \). As an action on \( \mathbb{B} \), we have \( D_w = \tilde{F}_w \).

In fact, some properties of \( D_w \) can be generalized to other affine types.

Lemma

For \( f, g \in \mathbb{B} \) and \( w \in \widetilde{S}_n \), we have

\[
D_w(fg) = \sum_{u,v} D_u(f)D_v(g).
\]

where \( \ell(w) = \ell(u) + \ell(v) \).
Question

Can we construct certain combinatorics similar to strong/weak tableaux for other types?

Related work:

- Pon introduced affine Stanley symmetric functions for the special orthogonal groups and Pieri factor generalizing $\rho_i$.
- Lam and Shimozono defined the analogue of the marked strong cover and weak cover for an arbitrary Kac-Moody algebra.


References

Pieri operators on the affine nilCoxeter algebra

Kostant and Kumar (1986)
The nil Hecke ring and cohomology of $G/P$ for a Kac-Moody group $G$.

Affine insertion and Pieri rules for the affine Grassmannian.
*Memoirs of the AMS* 208, no. 977.

S. Lee (2014)
Pieri rule for the affine flag variety.
arxiv.org/abs/1406.4246
Thank you!