Pieri Rules for Schur functions in superspace

Miles Eli Jones
joint work with Luc Lapointe

Universidad de Talca, Chile

July 9, 2015

Overview

Symmetric function theory

Overview

Symmetric function theory
Symmetric function theory IN SUPER SPACE!!!!!

Symmetric Function Theory

$$\mathbb{K}[z_1, \ldots, z_N]^{S_N}$$

1. partitions

$(5, 4, 2, 2, 1)$

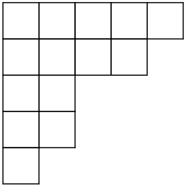
2. simple bases $(m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda, \ldots)$

3. other bases: Macdonald polynomials, Jack polynomials

Symmetric Function Theory

$$\mathbb{K}[z_1, \ldots, z_N]^{S_N}$$

1. partitions

$(5, 4, 2, 2, 1)$ 

2. simple bases $(m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda, \ldots)$

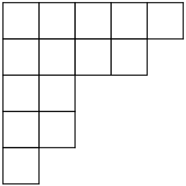
3. other bases: Macdonald polynomials, Jack polynomials

Symmetric Function Theory

$$\mathbb{K}[z_1, \ldots, z_N]^{S_N}$$

1. partitions

$(5, 4, 2, 2, 1)$



2. simple bases $(m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda, \ldots)$

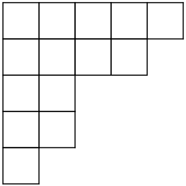
3. other bases: Macdonald polynomials, Jack polynomials

Symmetric Function Theory

$$\mathbb{K}[z_1, \ldots, z_N]^{S_N}$$

1. partitions

$(5, 4, 2, 2, 1)$



2. simple bases $(m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda, \ldots)$

3. other bases: Macdonald polynomials, Jack polynomials

Symmetric function theory

$$M_\lambda^{(q,t)}$$

Symmetric function theory

$$M_\lambda^{(q,t)}$$

$$\Big\downarrow \; {\scriptstyle q=t^\alpha,\; t\to 1}$$

Symmetric function theory

$$M_\lambda^{(q,t)} \xrightarrow[t\to 1]{q=t^\alpha} J_\lambda^{(\alpha)}$$

Symmetric function theory

$$M_\lambda^{(q,t)}$$

$$\Big\downarrow \; {\scriptstyle q=t^\alpha,\; t\to 1}$$

$$J_\lambda^{(\alpha)}$$

$$\Big\downarrow \; {\scriptstyle \alpha\to 1}$$

Symmetric function theory

$$
\begin{array}{c}
M_\lambda^{(q,t)} \\
\Big\downarrow {\scriptstyle q=t^\alpha,\ t\to 1} \\
J_\lambda^{(\alpha)} \\
\Big\downarrow {\scriptstyle \alpha\to 1} \\
s_\lambda
\end{array}
$$

Symmetric function theory

$M_\lambda^{(q,t)}$ COMBINATORICS

$\downarrow$ $q=t^\alpha$, $t\to 1$

$J_\lambda^{(\alpha)}$ PHYSICS

$\downarrow$ $\alpha\to 1$

s_λ combinatorics

Symmetric function theory

Common characterization:

- $M_\lambda^{(q,t)} = m_\lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\lambda^{(q,t)}, M_\mu^{(q,t)} \rangle = 0 \quad \text{if } \lambda \neq \mu$ (orthogonality)

Scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Symmetric function theory

Common characterization:

- $M_\lambda^{(q,t)} = m_\lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\lambda^{(q,t)}, M_\mu^{(q,t)} \rangle = 0 \quad \text{if } \lambda \neq \mu$ (orthogonality)

Scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Macdonald positivity

$$\tilde{M}_\lambda^{(q,t)} = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu \qquad \text{with} \qquad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$$

$K_{\mu\lambda}(1,1) =$ number of standard tableaux of shape μ

Macdonald positivity

$$\tilde{M}_\lambda^{(q,t)} = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu \qquad \text{with} \qquad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$$

$K_{\mu\lambda}(1,1) =$ number of standard tableaux of shape μ

Macdonald positivity

$$\tilde{M}^{(q,t)}_{(2,1)} = t\, s_{(3)} + (1 + qt)\, s_{(2,1)} + q\, s_{(1,1,1)}$$

1	2	3

1	2
3	

1	3
2	

1
2
3

Macdonald positivity

$$\tilde{M}^{(q,t)}_{(2,1)} = t\, s_{(3)} + (1 + qt)\, s_{(2,1)} + q\, s_{(1,1,1)}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Supersymmetry

2 types of particles in nature

bosons (integer spin: $0, 1, 2, \ldots$)

fermions (half integer spin: $1/2, 3/2, \ldots$)

$\Psi \longrightarrow \Psi$
exchange of two bosons

$\Psi \longrightarrow -\Psi$
exchange of two fermions
(*Pauli's exclusion principle*)

Supersymmetry

2 types of particles in nature

bosons (integer spin: $0, 1, 2, \ldots$)

fermions (half integer spin: $1/2, 3/2, \ldots$)

$\Psi \longrightarrow \Psi$

exchange of two bosons

$\Psi \longrightarrow -\Psi$

exchange of two fermions

(*Pauli's exclusion principle*)

Supersymmetry

2 types of particles in nature

bosons (integer spin: $0, 1, 2, \ldots$)

fermions (half integer spin: $1/2, 3/2, \ldots$)

$\Psi \longrightarrow \Psi$

exchange of two bosons

$\Psi \longrightarrow -\Psi$

exchange of two fermions

(*Pauli's exclusion principle*)

Supersymmetry

2 types of particles in nature

bosons (integer spin: $0, 1, 2, \ldots$)

fermions (half integer spin: $1/2, 3/2, \ldots$)

$$\Psi \longrightarrow \Psi$$

exchange of two bosons

$$\Psi \longrightarrow -\Psi$$

exchange of two fermions
(*Pauli's exclusion principle*)

A symmetric function theory in superspace

$$\mathbb{K}[z_1, \ldots, z_N, \theta_1, \ldots, \theta_N]^{S_N} \qquad \text{with} \qquad (\theta_i\theta_j = -\theta_i\theta_j \quad \text{and} \quad \theta_i^2 = 0)$$

<u>$N = 2$:</u> $(z_1 - z_2)\,\theta_1\theta_2$

Power sums: $p_1, p_2, \ldots, \tilde{p}_0, \tilde{p}_1, \ldots$

$$p_r = z_1^r + z_2^r + \cdots \qquad \text{and} \qquad \tilde{p}_k = \theta_1 z_1^k + \theta_2 z_2^k + \cdots$$

$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \qquad \text{and} \qquad \tilde{p}_i^2 = 0$$

A symmetric function theory in superspace

$$\mathbb{K}[z_1,\ldots,z_N,\theta_1,\ldots,\theta_N]^{S_N} \qquad \text{with} \qquad (\theta_i\theta_j = -\theta_i\theta_j \quad \text{and} \quad \theta_i^2 = 0)$$

$\underline{N=2:}$ $(z_1 - z_2)\,\theta_1\theta_2$

Power sums: $p_1, p_2, \ldots, \tilde{p}_0, \tilde{p}_1, \ldots$

$$p_r = z_1^r + z_2^r + \cdots \qquad \text{and} \qquad \tilde{p}_k = \theta_1 z_1^k + \theta_2 z_2^k + \cdots$$

$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \qquad \text{and} \qquad \tilde{p}_i^2 = 0$$

A symmetric function theory in superspace

$$\mathbb{K}[z_1,\ldots,z_N,\theta_1,\ldots,\theta_N]^{S_N} \qquad \text{with} \qquad (\theta_i\theta_j = -\theta_i\theta_j \quad \text{and} \quad \theta_i^2 = 0)$$

<u>$N = 2$:</u> $(z_1 - z_2)\,\theta_1\theta_2$

Power sums: $p_1, p_2, \ldots, \tilde{p}_0, \tilde{p}_1, \ldots$

$$p_r = z_1^r + z_2^r + \cdots \qquad \text{and} \qquad \tilde{p}_k = \theta_1 z_1^k + \theta_2 z_2^k + \cdots$$

$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \qquad \text{and} \qquad \tilde{p}_i^2 = 0$$

A symmetric function theory in superspace

$$\mathbb{K}[z_1,\ldots,z_N,\theta_1,\ldots,\theta_N]^{S_N} \qquad \text{with} \qquad (\theta_i\theta_j = -\theta_i\theta_j \quad \text{and} \quad \theta_i^2 = 0)$$

$\underline{N=2:}$ $(z_1 - z_2)\,\theta_1\theta_2$

Power sums: $p_1, p_2, \ldots, \tilde{p}_0, \tilde{p}_1, \ldots$

$$p_r = z_1^r + z_2^r + \cdots \qquad \text{and} \qquad \tilde{p}_k = \theta_1 z_1^k + \theta_2 z_2^k + \cdots$$

$$\tilde{p}_i\tilde{p}_j = -\tilde{p}_j\tilde{p}_i \qquad \text{and} \qquad \tilde{p}_i^2 = 0$$

Superpartitions

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \begin{cases} \Lambda^s \text{ is usual partition} \\ \Lambda^a \text{ has no repeated parts} \end{cases}$$

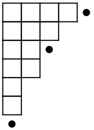
$$(4, 2, 0; 3, 2, 1, 1) \longleftrightarrow (4, 3, 2, 2, 1, 1, 0) \longleftrightarrow$$

simple bases $(m_\Lambda, p_\Lambda, e_\Lambda, h_\Lambda, s_\Lambda, \ldots)$

other bases: Macdonald polynomials and Jack polynomials in superspace.

Superpartitions

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \begin{cases} \Lambda^s \text{ is usual partition} \\ \Lambda^a \text{ has no repeated parts} \end{cases}$$

$(4, 2, 0; 3, 2, 1, 1) \longleftrightarrow (4, 3, 2, 2, 1, 1, 0) \longleftrightarrow$ 

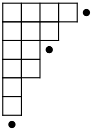
simple bases $(m_\Lambda, p_\Lambda, e_\Lambda, h_\Lambda, s_\Lambda, \ldots)$

other bases: Macdonald polynomials and Jack polynomials in superspace.

Superpartitions

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \begin{cases} \Lambda^s \text{ is usual partition} \\ \Lambda^a \text{ has no repeated parts} \end{cases}$$

$$(4, 2, 0; 3, 2, 1, 1) \longleftrightarrow (4, 3, 2, 2, 1, 1, 0) \longleftrightarrow$$



simple bases $(m_\Lambda, p_\Lambda, e_\Lambda, h_\Lambda, s_\Lambda, \ldots)$

other bases: Macdonald polynomials and Jack polynomials in superspace.

Symmetric function theory in superspace

Common characterization:

- $M_\Lambda^{(q,t)} = m_\Lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\Lambda^{(q,t)}, M_\Omega^{(q,t)} \rangle = 0 \quad \text{if } \Lambda \neq \Omega$ (orthogonality)

Scalar product:

$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s} \prod_i \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}}$$

$$\Lambda = (\Lambda^a; \Lambda^s) \qquad \text{and} \qquad p_\Lambda = \tilde{p}_{\Lambda^a} p_{\Lambda^s}$$

Symmetric function theory in superspace

Common characterization:

- $M_\Lambda^{(q,t)} = m_\Lambda + \text{smaller terms}$ (triangularity)
- $\langle M_\Lambda^{(q,t)}, M_\Omega^{(q,t)} \rangle = 0 \quad \text{if } \Lambda \neq \Omega$ (orthogonality)

Scalar product:

$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s} \prod_i \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}}$$

$\Lambda = (\Lambda^a; \Lambda^s)$ and $p_\Lambda = \tilde{p}_{\Lambda^a} \, p_{\Lambda^s}$

Symmetric function theory in superspace

$$M_\Lambda^{(q,t)}$$

$$\downarrow$$

Symmetric function theory in superspace

$$M_\Lambda^{(q,t)} \xrightarrow[t\to 1]{q=t^\alpha} J_\Lambda^{(\alpha)}$$

Symmetric function theory in superspace

$$M_\Lambda^{(q,t)}$$

$$\Big\downarrow \; {\scriptstyle q=t^\alpha,\ t\to 1}$$

$$J_\Lambda^{(\alpha)}$$

$$\Big\downarrow \; {\scriptstyle \alpha\to 1}$$

$$s_\Lambda^1$$

Symmetric function theory in superspace

$$M_\Lambda^{(q,t)}$$ COMBINATORICS

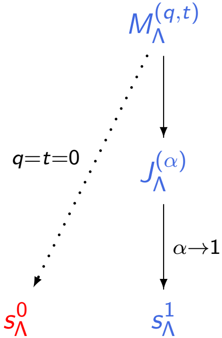
$$\Big\downarrow {\scriptstyle q=t^\alpha,\ t\to 1}$$

$$J_\Lambda^{(\alpha)}$$ PHYSICS

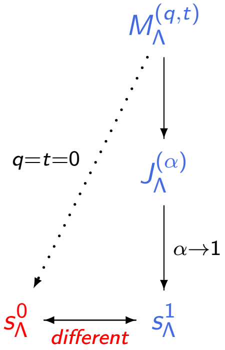
$$\Big\downarrow {\scriptstyle \alpha\to 1}$$

$$s_\Lambda^1$$ yucky combinatorics

Symmetric function theory in superspace



Symmetric function theory in superspace



$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s}(q,t)$$

Symmetric function theory in superspace

$M_{\Lambda}^{(q,t)}$

$q=t=0$

s_{Λ}^{0}

Macdonald positivity conjecture in superspace!!

$$\tilde{M}_{\Lambda}^{(q,t)} = \sum_{\Omega} K_{\Omega\Lambda}(q,t)\, s_{\Omega}^{0} \qquad \text{with} \qquad K_{\Omega\Lambda}(q,t) \in \mathbb{N}[q,t]???$$

Macdonald positivity conjecture in superspace

$$\tilde{M}^{(q,t)}_{\begin{ytableau} \ & \none[\bullet] \\ \ \end{ytableau}} = t\, s^{0}_{\begin{ytableau} \ & \ & \none[\bullet] \end{ytableau}} + s^{0}_{\begin{ytableau} \ & \none[\bullet] \\ \ \end{ytableau}} + qt\, s^{0}_{\begin{ytableau} \ & \ \\ \none[\bullet] \end{ytableau}} + q\, s^{0}_{\begin{ytableau} \ \\ \ \\ \none[\bullet] \end{ytableau}}$$

$$\tilde{M}^{(q,t)}_{\begin{ytableau} \ & \ \\ \ \end{ytableau}} = t\, s_{\begin{ytableau} \ & \ & \ \end{ytableau}} + (1+qt)\, s_{\begin{ytableau} \ & \ \\ \ \end{ytableau}} + q\, s_{\begin{ytableau} \ \\ \ \\ \ \end{ytableau}}$$

Refinement of the original problem!!

Macdonald positivity conjecture in superspace

$$\tilde{M}^{(q,t)}_{\ydiagram{1,1}\,\bullet} = t\, s^0_{\ydiagram{2}\,\bullet} + s^0_{\ydiagram{1,1}\,\bullet} + qt\, s^0_{\ydiagram{2}\atop\bullet} + q\, s^0_{\ydiagram{1,1}\atop\bullet}$$

$$\tilde{M}^{(q,t)}_{\ydiagram{2,1}} = t\, s_{\ydiagram{3}} + (1+qt)\, s_{\ydiagram{2,1}} + q\, s_{\ydiagram{1,1,1}}$$

Refinement of the original problem!!

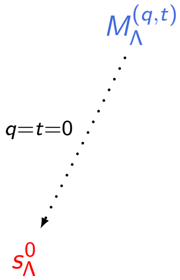
Macdonald positivity conjecture in superspace

$$\tilde{M}^{(q,t)}_{(1;1)} = t\, s^{0}_{(2;)} + s^{0}_{(1;1)} + qt\, s^{0}_{(0;2)} + q\, s^{0}_{(0;1,1)}$$

$$\tilde{M}^{(q,t)}_{(2,1)} = t\, s_{(3)} + s_{(2,1)} + qt\, s_{(2,1)} + q\, s_{(1,1,1)}$$

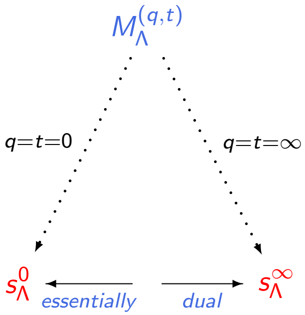
Refinement of the original problem!!

Symmetric function theory in superspace



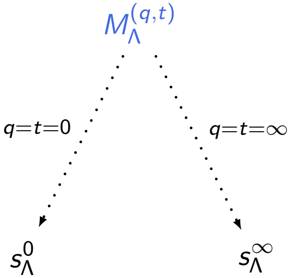
$$\langle p_\Lambda, p_\Omega \rangle_{qt} = \delta_{\Lambda\Omega} q^{|\Lambda^a|} z_{\Lambda^s}(q,t)$$

Symmetric function theory in superspace



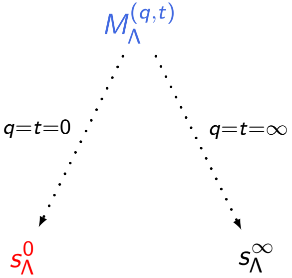
Symmetric function theory in superspace

Pieri rules, tableaux generating functions (monomial expansions), Cauchy identities (RSK).

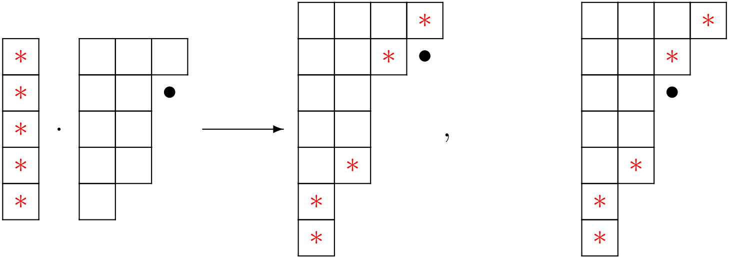


Symmetric function theory in superspace

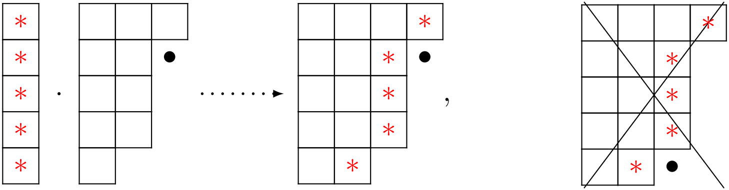
Mathieu
Blondeau-Fournier
Pieri Rule



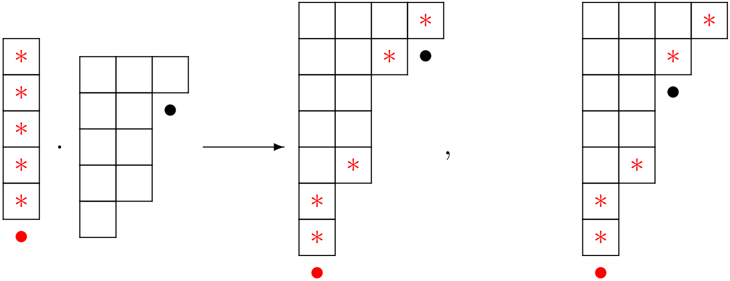
Pieri Rule for $e_r \cdot s^0_\Lambda$



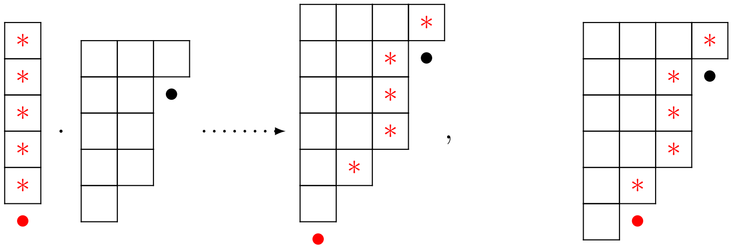
Pieri Rule for $e_r \cdot s^0_\Lambda$



Pieri Rule for $\tilde{e}_r \cdot s^0_\Lambda$



Pieri Rule for $\tilde{e}_r \cdot s^0_\Lambda$



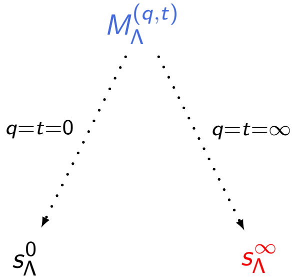
Monomial expansions

$$s_\Lambda^0 \xleftarrow[\textit{essentially}]{} \quad \xrightarrow[\textit{dual}]{} s_\Lambda^\infty$$

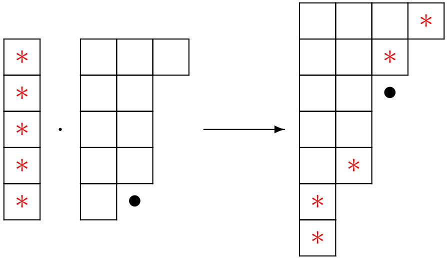
$$s_\Lambda^\infty = \sum_{T \in Tab_\Lambda^0} (x\theta)^T$$

Symmetric function theory in superspace

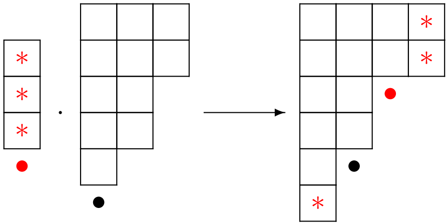
Lapointte
Preville-Ratelle
MJ
Pieri Rule for s^{∞}_{Λ}.



Pieri Rule for $e_r \cdot s_\Lambda^\infty$



Pieri Rule for $\tilde{e}_r \cdot s_\Lambda^\infty$



Monomial expansions

$$s_\Lambda^0 \xleftarrow[\textit{essentially}]{} \xrightarrow[\textit{dual}]{} s_\Lambda^\infty$$

$$s_\Lambda^\infty = \sum_{T \in Tab_\Lambda^0} (x\theta)^T$$

$$s_\Lambda^0 = \sum_{T \in Tab_\Lambda^\infty} (x\theta)^T$$

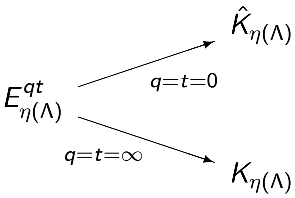
Cauchy Identity

$$\sum_\Lambda s_\Lambda^0(x,\theta) s_{\Lambda'}^\infty(y,\phi) = \prod_{i,j}(1 + x_i y_j + \theta_i \phi_j)$$

Pieri Rules proofs

$$M_\Lambda^{qt} = \mathcal{O}^{qt} E_{\eta(\Lambda)}^{qt} \theta_1 \ldots \theta_m + \text{other terms}_{(\text{symmetrization})}$$

$\eta(\Lambda)$ is a composition based on Λ
$E_{\eta(\Lambda)}^{qt}$ is a non-symmetric Macdonald polynomial.
$\mathcal{O}^{qt}$ is some operator.

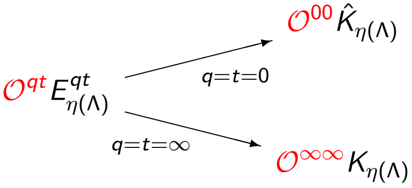


Key polynomials

Pieri Rules proofs

$$M_\Lambda^{qt} = \mathcal{O}^{qt} E^{qt}_{\eta(\Lambda)} \theta_1 \dots \theta_m + \text{other terms}_{(\text{symmetrization})}$$

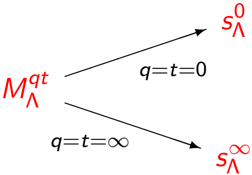
$\eta(\Lambda)$ is a composition based on Λ
$E^{qt}_{\eta(\Lambda)}$ is a non-symmetric Macdonald polynomial.
$\mathcal{O}^{qt}$ is some operator.



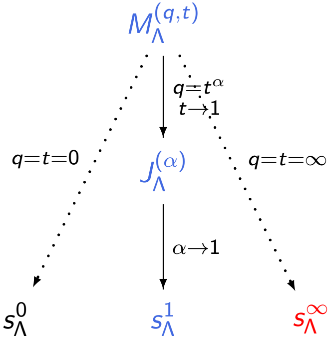
Pieri Rules proofs

$$M^{qt}_{\Lambda} = \mathcal{O}^{qt} E^{qt}_{\eta(\Lambda)} \theta_1 \dots \theta_m + \text{other terms}_{(\text{symmetrization})}$$

$\eta(\Lambda)$ is a composition based on Λ
$E^{qt}_{\eta(\Lambda)}$ is a non-symmetric Macdonald polynomial.
$\mathcal{O}^{qt}$ is some operator.



Symmetric function theory in superspace



COMBINATORICS

PHYSICS

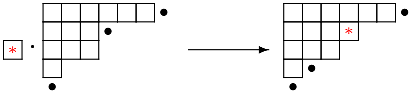
$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_1 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.

$$\frac{3\alpha(5\alpha+2)}{(3\alpha+2)^2(5\alpha+3)}$$

$J_\Lambda^{(\alpha)}$

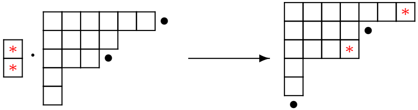
The Pieri Rules for $e_1 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.



$$\frac{3\alpha(5\alpha+2)}{(3\alpha+2)^2(5\alpha+3)}$$

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_2 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are quadratic factors

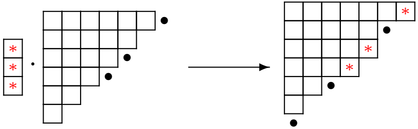


$$-\frac{2\alpha^3(3\alpha^2+\alpha-1)}{(6\alpha+5)(7\alpha+5)(\alpha+1)(\alpha+2)(3\alpha+1)(2\alpha+1)}$$

Sum of 2 terms

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_3 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are degree 6 factors!!!!!!!!

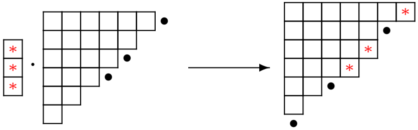


$$\frac{1}{1152}\,\frac{\alpha^4(2\alpha+3)(3\alpha+4)(416\alpha^6+2000\alpha^5+3484\alpha^4+2608\alpha^3+559\alpha^2-256\alpha-108)}{(4\alpha+3)(5\alpha+4)(7\alpha+6)(2\alpha+1)(\alpha+1)^{10}}$$

Sum of 6 terms???????

$J_\Lambda^{(\alpha)}$

The Pieri Rules for $e_3 J_\Lambda^{(\alpha)}$ are quotients of linear factors in α.
Sometimes there are degree 6 factors!!!!!!!!



$$\frac{1}{1152}\,\frac{\alpha^4(2\alpha+3)(3\alpha+4)(416\alpha^6+2000\alpha^5+3484\alpha^4+2608\alpha^3+559\alpha^2-256\alpha-108)}{(4\alpha+3)(5\alpha+4)(7\alpha+6)(2\alpha+1)(\alpha+1)^{10}}$$

Sum of 7 terms!!!!!!!!!!!!!

Alternating Sign Matrices!!!!!

$$1, 1, 2, 7, 42, 429, \ldots$$

Square Ice!!!!

Sum corresponds to partition function of square ice!!!!!!!

$$
\begin{array}{ccccccccccccccccc}
H & - & O & & H & - & O & - & H & & O & - & H & & O & - & H \\
 & & | & & & & & & & & | & & & & | & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & & & & & | & & & & & & & & & & \\
H & - & O & - & H & & O & & H & - & O & - & H & & O & - & H \\
 & & & & & & | & & & & & & & & | & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & | & & & & & & & & | & & & & & & \\
H & - & O & & H & - & O & & H & - & O & & H & - & O & - & H \\
 & & & & & & | & & & & & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & | & & & & & & & & | & & & & | & & \\
H & - & O & & H & - & O & - & H & & O & - & H & & O & - & H
\end{array}
$$

Thank you