The Freeness of Ish Arrangements

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July 9, 2015

First of all, we define three types of arrangements in the ℓ-dimensional Euclidean space E with coordinates $x_1, \ldots, x_\ell$:

- Coxeter arrangement Cox(ℓ)
- Shi arrangement Shi(ℓ)
- Ish arrangement Ish(ℓ)

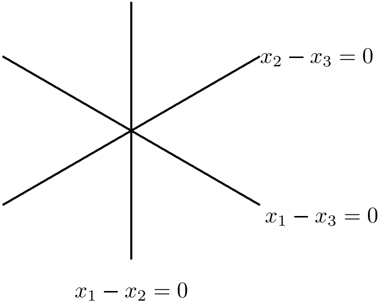
Three types of arrangement

Coxeter arrangement

$$\mathrm{Cox}(\ell) = \left\{ \{x_i - x_j = 0\} \,\middle|\, 1 \leq i < j \leq \ell \right\}$$

the **Coxeter arrangement** of the type $A_{\ell-1}$

Cox(3)

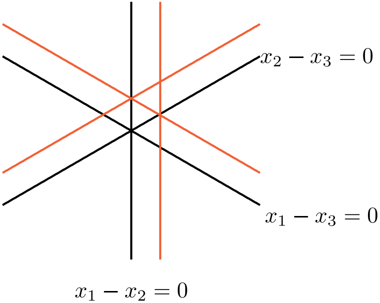


Shi arrangement

$$\mathrm{Shi}(\ell) = \mathrm{Cox}(\ell) \cup \left\{ \{x_i - x_j = 1\} \,\middle|\, 1 \leq i < j \leq \ell \right\}$$

the **Shi arrangement** of the type $A_{\ell-1}$

Shi(3)



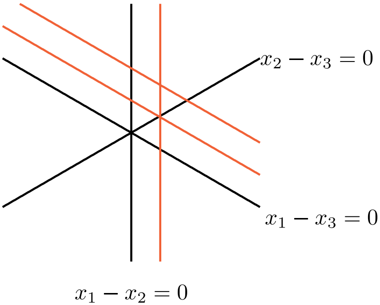
Three types of arrangement

Ish arrangement

$$\mathrm{Ish}(\ell) = \mathrm{Cox}(\ell) \cup \left\{ \{x_1 - x_j = i\} \,\middle|\, 1 \leq i < j \leq \ell \right\}$$

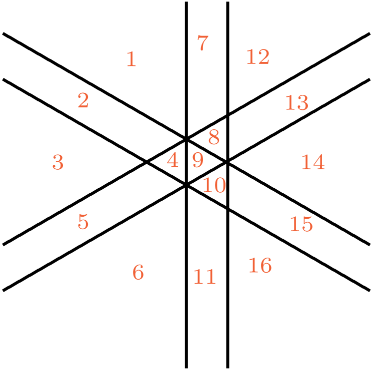
the **Ish arrangement** (of the type $A_{\ell-1}$)

Ish(3)

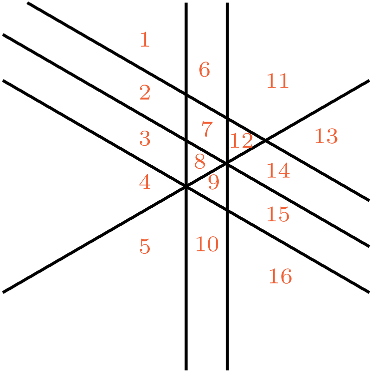


The number of chambers

Shi(ℓ) and Ish(ℓ) have the same number of chambers



16 chambers



16 chambers

The number of chambers

Definition

Let $\mathcal{A}$ be an arrangement. Define the ***intersection poset*** $L(\mathcal{A})$ *by*

$$L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \neq \emptyset \;\middle|\; \mathcal{B} \subset \mathcal{A} \right\}, \quad X \leq Y \Leftrightarrow Y \subseteq X,$$

the ***Möbius function*** $\mu_{\mathcal{A}} : L(\mathcal{A}) \to \mathbb{Z}$ *by*

$$\mu_{\mathcal{A}}(E) = 1, \quad \mu_{\mathcal{A}}(X) = -\sum_{\substack{Y \in L(\mathcal{A}) \\ E \leq Y < X}} \mu_{\mathcal{A}}(Y) \ \ (X \neq E).$$

and the ***characteristic polynomial*** $\chi(\mathcal{A}, t)$ *by*

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu_{\mathcal{A}}(X)\, t^{\dim X}.$$

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The number of chambers

Theorem (T. Zaslavsky, 1975)

Let $\mathcal{A}$ be an arrangement in $E = \mathbb{R}^\ell$. Then the number of chambers of $\mathcal{A}$ is $|\chi(\mathcal{A}, -1)|$.

Theorem (D. Armstrong, 2011)

The characteristic polynomial of the Shi arrangement is equal to the characteristic polynomial of the Ish arrangement:

$$\chi(\mathrm{Shi}(\ell), t) = \chi(\mathrm{Ish}(\ell), t) = t(t-\ell)^{\ell-1}.$$

Hence the number of chambers of the Shi arrangement and Ish arrangement are the same.

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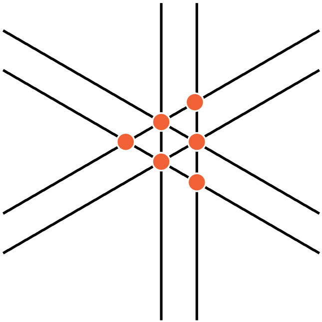
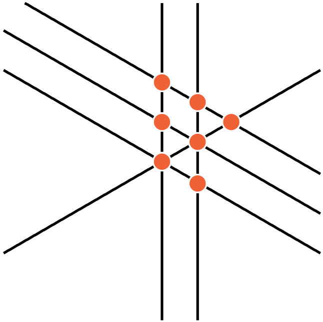
The intersection posets of Shi and Ish

Remark

Shi(ℓ) *and* Ish(ℓ) *share the same characteristic polynomial, although* $L(\mathrm{Shi}(\ell)) \not\cong L(\mathrm{Ish}(\ell))$.



6 intersection points



7 intersection points

Deleted Shi and Ish arrangements

Let $K_\ell := \{(i,j) \mid 1 \leq i < j \leq \ell\}$ be the edge set of the complete graph with vertices $\{1, 2, \ldots, \ell\}$.

Definition

Let $G \subseteq K_\ell$ be a subgraph. Define the ***deleted Shi and Ish arrangements*** *as follows:*

$$\mathrm{Shi}(G) := \mathrm{Cox}(\ell) \cup \left\{\{x_i - x_j = 1\} \,\middle|\, (i,j) \in G\right\}$$

$$\mathrm{Ish}(G) := \mathrm{Cox}(\ell) \cup \left\{\{x_1 - x_j = i\} \,\middle|\, (i,j) \in G\right\}$$

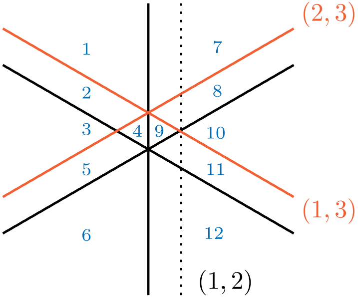
e.g.

$$\mathrm{Shi}(K_\ell) = \mathrm{Shi}(\ell), \quad \mathrm{Shi}(\emptyset) = \mathrm{Cox}(\ell)$$
$$\mathrm{Ish}(K_\ell) = \mathrm{Ish}(\ell), \quad \mathrm{Ish}(\emptyset) = \mathrm{Cox}(\ell)$$

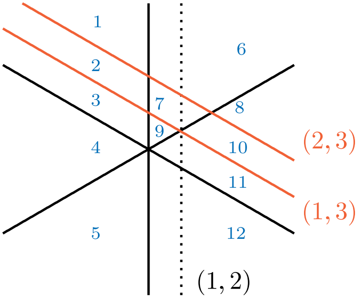
We have the following correspondence between Shi(G) and Ish(G):

$$\begin{array}{ccccc} \mathrm{Shi}(G) \setminus \mathrm{Cox}(\ell) & \leftrightarrow & \mathrm{G} & \leftrightarrow & \mathrm{Ish}(G) \setminus \mathrm{Cox}(\ell) \\ \Cup & & \Cup & & \Cup \\ \{x_i - x_j = 1\} & \leftrightarrow & (i,j) & \leftrightarrow & \{x_1 - x_j = i\} \end{array}$$

$\underline{G = \{(1,3),(2,3)\} \subseteq K_3}$



12 chambers



12 chambers

Deleted Shi and Ish arrangements

Theorem (D. Armstrong and B. Rhoades, 2012)

The characteristic polynomial of $\mathrm{Shi}(G)$ *is equal to the characteristic polynomial of* $\mathrm{Ish}(G)$*:*

$$\chi(\mathrm{Shi}(G), t) = \chi(\mathrm{Ish}(G), t).$$

The freeness of arrangements

From now on, we will consider the problem of whether Ish(G) is a free arrangement or not.

Definition

Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$. Define the ***derivation module*** $\mathrm{Der}_{\mathbb{K}}(S)$ *by*

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\theta : S \to S \mid \theta : \mathbb{K}\text{-linear}, \\ \text{for any } f, g \in S,\ \theta(fg) = f\theta(g) + \theta(f)g\}.$$

Then $\mathrm{Der}_{\mathbb{K}}(S)$ *is an S-module.*

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Then $\mathrm{Der}_{\mathbb{K}}(S)$ *is an* S*-module.*

The freeness of arrangements

Definition

*For a hyperplane H, choose $\alpha_H \in S$ such that $\{\alpha_H = 0\} = H$. For an arrangement $\mathcal{A}$ of hyperplanes, the polynomial $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H \in S$ is called a **defining polynomial**. For a central arrangement $\mathcal{A}$, (i.e. $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$) define the **logarithmic derivation module** $D(\mathcal{A})$ of $\mathcal{A}$ by*

$$\begin{aligned} D(\mathcal{A}) &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\} \\ &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for any } H \in \mathcal{A}\}. \end{aligned}$$

*We say that $\mathcal{A}$ is **free**, if $D(\mathcal{A})$ is a free S-module.*
*When $\mathcal{A}$ is free, there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$, and the multiset $\exp(\mathcal{A}) = (\deg \theta_1, \ldots, \deg \theta_\ell)$ is called the **exponent** of $\mathcal{A}$.*

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The freeness of arrangements

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An arrangement $\mathcal{A}$ is called ***supersolvable*** *if there exists a filtration $\mathcal{A} = \mathcal{A}_\ell \supseteq \mathcal{A}_{\ell-1} \supseteq \cdots \supseteq \mathcal{A}_1$ such that*

1. $\operatorname{codim} \bigcap_{H \in \mathcal{A}_i} H = i \ \ (i = 1, 2, \ldots, \ell)$
2. *For any $H, H' \in \mathcal{A}_i$ with $H \neq H'$, there exists some $H'' \in \mathcal{A}_{i-1}$ such that $H \cap H' \subseteq H''$.*

Cox(ℓ) is supersolvable

$\mathrm{Cox}(k) = \left\{ \{x_i - x_j = 0\} \,\middle|\, 1 \leq i < j \leq k \right\}$ $(2 \leq k \leq \ell)$ can be regarded as arrangements in $\mathbb{R}^{\ell-1}$. The filtration

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The freeness of arrangements

Theorem (Addition theorem, H. Terao, 1980)

Let $\mathcal{A}$ be a central arrangement and $H_0 \in \mathcal{A}$. If $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ and $\mathcal{A}'' := \{H_0 \cap H \mid H \in \mathcal{A}'\}$ are free and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$, then $\mathcal{A}$ is free.

Definition

*Define the **inductive freeness** by the following:*

1. *The empty arrangement is inductively free.*
2. *$\mathcal{A}$ is inductively free if there exists $H_0 \in \mathcal{A}$ such that $\mathcal{A}'$ and $\mathcal{A}''$ are inductively free and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$.*

Theorem

supersolvable $\Rightarrow$ inductively free $\Rightarrow$ free

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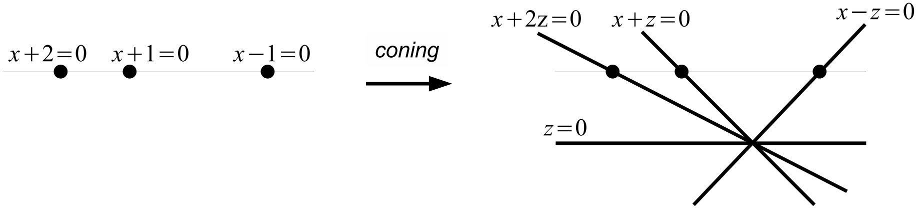
The coning

Definition

Let $\mathcal{A}$ be an affine arrangement in E and $E^* = \langle x_1, \ldots, x_\ell \rangle_{\mathbb{R}}$. Let W be a vector space such that $W^* = \langle x_1, \ldots, x_\ell, z \rangle_{\mathbb{R}}$. Then we define a central arrangement $\mathbf{c}\mathcal{A}$ in W by

$$Q(\mathbf{c}\mathcal{A}) = z \cdot z^{\deg Q(\mathcal{A})} Q(\mathcal{A}) \left(\frac{x_1}{z}, \ldots, \frac{x_\ell}{z} \right).$$

$\mathbf{c}\mathcal{A}$ *is called the **cone** over* $\mathcal{A}$.



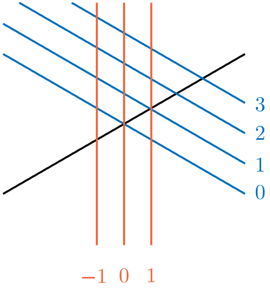
N-Ish arrangement

Definition

*Let $N = (N_2, N_3, \ldots, N_\ell)$ with finite sets $N_j \subseteq \mathbb{R}$ for $2 \le j \le \ell$. Define the **N-Ish arrangement** $\mathrm{Ish}(N)$ by*

$$\mathrm{Ish}(N) = \left\{\{x_i - x_j = 0\} \,\middle|\, 2 \le i < j \le \ell\right\} \cup \left\{\{x_1 - x_j = a\} \,\middle|\, 2 \le j \le \ell, a \in N_j\right\}.$$

$\mathrm{Ish}(N) \quad N = (\{-1, 0, 1\}, \{0, 1, 2, 3\})$



N-Ish arrangement

The class of N-Ish arrangements includes the deleted Ish arrangements.

$$N_j = \{0, 1, \ldots, j-1\} \Rightarrow \mathrm{Ish}(N) = \mathrm{Ish}(\ell)$$

$$N_j = \{0\} \cup \{i \mid (i, j) \in G\} \Rightarrow \mathrm{Ish}(N) = \mathrm{Ish}(G)$$

Definition

*$N = (N_2, \ldots, N_\ell)$ is a **nest** if there exists a permutation w of $\{2, 3, \ldots, \ell\}$ such that $N_{w(2)} \subseteq N_{w(3)} \subseteq \cdots \subseteq N_{w(\ell)}$*

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N-Ish arrangement

Theorem (Abe, S, Tsujie)

For N-Ish arrangements, the following are equivalent:

1. *N is a nest,*
2. **cIsh**(N) *is supersolvable,*
3. **cIsh**(N) *is inductively free,*
4. **cIsh**(N) *is free.*

(① ⇒ ②) Let $N_2 \supseteq N_3 \supseteq \cdots \supseteq N_\ell$,

$$\mathcal{I}_i := \left\{ \{x_1 - x_j = az\} \,\middle|\, 2 \le j \le i, a \in N_j \right\} \cup \left\{ \{x_j - x_k = 0\} \,\middle|\, 2 \le j < k \le i \right\} \cup \left\{ \{z = 0\} \right\}.$$

Then $\mathbf{cIsh}(N) = \mathcal{I}_\ell \supseteq \mathcal{I}_{\ell-1} \supseteq \cdots \supseteq \mathcal{I}_1$.

Corollary

cIsh(ℓ) *is a free arrangement.*

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(① ⇒ ②) Let $N_2 \supseteq N_3 \supseteq \cdots \supseteq N_\ell$,

$$\mathcal{I}_i := \left\{\{x_1 - x_j = az\} \,\middle|\, 2 \le j \le i, a \in N_j\right\} \cup \left\{\{x_j - x_k = 0\} \,\middle|\, 2 \le j < k \le i\right\} \cup \left\{\{z = 0\}\right\}.$$

Then $\mathbf{cIsh}(N) = \mathcal{I}_\ell \supseteq \mathcal{I}_{\ell-1} \supseteq \cdots \supseteq \mathcal{I}_1$.

Corollary

cIsh(ℓ) *is a free arrangement.*

N-Ish arrangement

Theorem (Abe, S, Tsujie)

Suppose that $N_2 \subseteq \cdots \subseteq N_\ell$. *Define homogeneous derivations* $\theta_0, \theta_1, \ldots, \theta_\ell$ *by*

$$\theta_0 := \sum_{i=1}^{\ell} \partial x_i, \qquad \theta_1 = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i} + z\frac{\partial}{\partial z},$$

$$\theta_k := \sum_{s=2}^{k} \left(\prod_{a \in N_k} (x_1 - x_s - az) \prod_{t=k+1}^{\ell} (x_s - x_t) \right) \frac{\partial}{\partial x_s} \quad (2 \leq k \leq \ell).$$

Then $\theta_0, \theta_1, \ldots, \theta_\ell$ *form a basis for* $D(\mathbf{c}\mathrm{Ish}(N))$. *In particular, the exponents are given by*

$$\exp(\mathbf{c}\mathrm{Ish}(N)) = (0,\ 1,\ |N_2| + \ell - 2,\ |N_3| + \ell - 3,\ \ldots\ ,\ |N_\ell|).$$

The characteristic polynomial of the Ish arrangement

Theorem (Factorization Theorem, H. Terao, 1981)

Let $\mathcal{A}$ be a free arrangement with $\exp(\mathcal{A}) = \{d_1, \ldots, d_\ell\}$. Then the characteristic polynomial of $\mathcal{A}$ factors as $\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(t - d_i)$.

The characteristic polynomial $\chi(\mathrm{Ish}(\ell), t)$

$\mathrm{Ish}(\ell)$ is $\mathrm{Ish}(N)$ with $N_j = \{0, 1, \ldots, j-1\}$.
Since $|N_j| = j$,

$$\begin{aligned}\exp(\mathbf{c}\mathrm{Ish}(\ell)) &= (0,\ 1,\ |N_2| + \ell - 2,\ |N_3| + \ell - 3,\ \ldots\ ,\ |N_\ell|) \\ &= (0,\ 1,\ \ell,\ \ldots\ ,\ \ell).\end{aligned}$$

Hence $\chi(\mathbf{c}\mathrm{Ish}(\ell), t) = t(t-1)(t-\ell)^{\ell-1}$. Since $\chi(\mathbf{c}\mathcal{A}, t) = (t-1)\chi(\mathcal{A}, t)$,

$$\chi(\mathrm{Ish}(\ell), t) = t(t-\ell)^{\ell-1}.$$

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$$\chi(\mathrm{Ish}(\ell), t) = t(t-\ell)^{\ell-1}.$$

Shi(G) and Ish(G)

Theorem (C. A. Athanasiadis, 1998)

Let $G \subseteq K_\ell$. Then Shi(G) *is free if and only if the following condition* (A) *is satisfied:*

(A) *There exists a permutation w of* $\{1, 2, \ldots, \ell\}$ *such that* $w^{-1}G \subseteq K_\ell$ *and*
$1 \leq i < j < k \leq \ell,\ (i, j) \in w^{-1}G \Rightarrow (i, k) \in w^{-1}G.$

For $G \subseteq K_\ell$, define $N_G = (N_2, \ldots, N_\ell)$ by

$$N_j = \{0\} \cup \{i \mid (i,j) \in G\}.$$

Then $\mathrm{Ish}(G) = \mathrm{Ish}(N_G)$.

Lemma

G satisfies the condition $(A) \Leftrightarrow N_G$ is a nest

$\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$

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Lemma

G satisfies the condition $(A) \Leftrightarrow N_G$ is a nest

Shi(G) **and** Ish(G)

Theorem (Abe, S, Tsujie)

Let $G \subseteq K_\ell$. Then the following are equivalent.

1. *G satisfies (A)*
2. *N_G is a nest*
3. **c**Shi(G) *is free*
4. **c**Ish(G) *is free*

Problem

Problem

Is there an isomorphism between $D(\mathbf{c}\mathrm{Shi}(G))$ *and* $D(\mathbf{c}\mathrm{Ish}(G))$ *when they are not free?*

Thank you very much
for your attention!