Walks in the Quarter Plane with Multiple Steps

Manuel Kauers and Rika Yatchak

Institute for Algebra
Johannes Kepler Universität

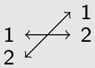
July 9, 2015

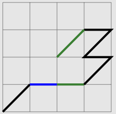
We consider 2D *lattice walks* in the positive quadrant $(\mathbb{Z}_{\geq 0})^2$ with *small steps* $s \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ and multiplicities.

We consider 2D *lattice walks* in the positive quadrant $(\mathbb{Z}_{\geq 0})^2$ with *small steps* $s \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ and multiplicities.
A *model* is a multiset of admissible steps s.

We consider 2D *lattice walks* in the positive quadrant $(\mathbb{Z}_{\geq 0})^2$ with *small steps* $s \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ and multiplicities. A *model* is a multiset of admissible steps s.

Example

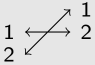


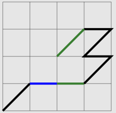


$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

We consider 2D *lattice walks* in the positive quadrant $(\mathbb{Z}_{\geq 0})^2$ with *small steps* $s \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ and multiplicities. A *model* is a multiset of admissible steps s.

Example





$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

Main question: given a model M, what is the nature of its generating function

$$F_M(x,y,t) = \sum_{n=0}^{\infty} \sum_{i,j} f_{i,j,n} x^i y^j t^n$$

Is it algebraic? D-finite? Something else?

A bit of history

A bit of history

Fayolle, Iasnogorodski, Malyshev: *Random Walks in the Quarter-Plane*

A bit of history

Fayolle, Iasnogorodski, Malyshev: *Random Walks in the Quarter-Plane* : aka the “Yellow book”

A bit of history

Fayolle, Iasnogorodski, Malyshev: *Random Walks in the Quarter-Plane* : aka the “Yellow book”

Bousquet-Mélou and Mishna: *Walks with Small Steps in the Quarter Plane*

Recall:

A power series $F(x, y, t) \in \mathbb{Q}[[x, y, t]]$ is:

Recall:

A power series $F(x, y, t) \in \mathbb{Q}[[x, y, t]]$ is:

- *algebraic* over $\mathbb{Q}(x, y, t)$ if there are polynomials $p_0, \ldots, p_n \in \mathbb{Q}[x, y, t]$ such that there is a nontrivial relation of the form

$$p_n F(x, y, t)^n + p_{n-1} F(x, y, t)^{n-1} + \cdots + p_0 = 0$$

Recall:

A power series $F(x, y, t) \in \mathbb{Q}[[x, y, t]]$ is:

- *algebraic* over $\mathbb{Q}(x, y, t)$ if there are polynomials $p_0, \ldots, p_n \in \mathbb{Q}[x, y, t]$ such that there is a nontrivial relation of the form

$$p_n F(x, y, t)^n + p_{n-1} F(x, y, t)^{n-1} + \cdots + p_0 = 0$$

- *D-finite* over $\mathbb{Q}(x, y, t)$ if the $\mathbb{Q}(x, y, t)$-VS spanned by all partial derivatives of $F(x, y, t)$ has finite dimension

Recall:

A power series $F(x, y, t) \in \mathbb{Q}[[x, y, t]]$ is:

- *algebraic* over $\mathbb{Q}(x, y, t)$ if there are polynomials $p_0, \dots, p_n \in \mathbb{Q}[x, y, t]$ such that there is a nontrivial relation of the form

$$p_n F(x, y, t)^n + p_{n-1} F(x, y, t)^{n-1} + \cdots + p_0 = 0$$

- *D-finite* over $\mathbb{Q}(x, y, t)$ if F satisfies a nontrivial linear DE for each $x_i \in \{x, y, t\}$ with coefficients in $\mathbb{Q}[x, y, z]$

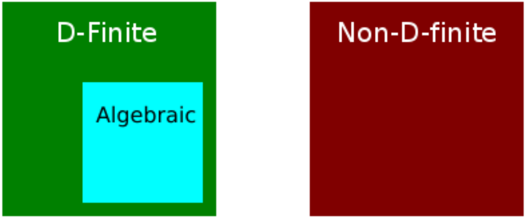
Recall:

A power series $F(x, y, t) \in \mathbb{Q}[[x, y, t]]$ is:

- *algebraic* over $\mathbb{Q}(x, y, t)$ if there are polynomials $p_0, \ldots, p_n \in \mathbb{Q}[x, y, t]$ such that there is a nontrivial relation of the form

$$p_n F(x, y, t)^n + p_{n-1} F(x, y, t)^{n-1} + \cdots + p_0 = 0$$

- *D-finite* over $\mathbb{Q}(x, y, t)$ if F satisfies a nontrivial linear DE for each $x_i \in \{x, y, t\}$ with coefficients in $\mathbb{Q}[x, y, z]$



Why care about algebraicity and D-finiteness?

Why care about algebraicity and D-finiteness?

Heuristically: we know these objects are “well-behaved”

Why care about algebraicity and D-finiteness?

Heuristically: we know these objects are “well-behaved”

Nice asymptotics:

Why care about algebraicity and D-finiteness?

Heuristically: we know these objects are "well-behaved"

Nice asymptotics:

Quadrant walks: $f_n \sim K \cdot \rho^n \cdot n^\alpha$ for some constants K, ρ, α

Why care about algebraicity and D-finiteness?

Heuristically: we know these objects are "well-behaved"

Nice asymptotics:

Quadrant walks: $f_n \sim K \cdot \rho^n \cdot n^\alpha$ for some constants K, ρ, α

Note that D-finiteness heavily depends on the model

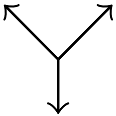
Why care about algebraicity and D-finiteness?

Heuristically: we know these objects are "well-behaved"

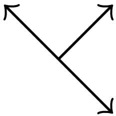
Nice asymptotics:

Quadrant walks: $f_n \sim K \cdot \rho^n \cdot n^\alpha$ for some constants K, ρ, α

Note that D-finiteness heavily depends on the model



D-finite GF



non D-finite GF

What's new?

What's new?

A classification of "interesting" D-finite and algebraic models with "small steps" in the quarter plane, with multiplicities.

What's new?

A classification of "interesting" D-finite and algebraic models with "small steps" in the quarter plane, with multiplicities.

Main methods: group of the model, Gröbner basis techniques, kernel method, orbit sums, half orbit sums, *guessing*

“Interesting” models

"Interesting" models

A model is interesting if it is:

"Interesting" models

A model is interesting if it is:

- Not equivalent to a half plane model

“Interesting” models

A model is interesting if it is:

- Not equivalent to a half plane model
- Not equivalent to some “interesting” model by reflection about the diagonal $x = y$

"Interesting" models

A model is interesting if it is:

- Not equivalent to a half plane model
- Not equivalent to some "interesting" model by reflection about the diagonal $x = y$
- Not equivalent to λS for some "interesting" model S, with $\lambda \neq 0$.

"Interesting" models

A model is interesting if it is:

- Not equivalent to a half plane model
- Not equivalent to some "interesting" model by reflection about the diagonal $x = y$
- Not equivalent to λS for some "interesting" model S, with $\lambda \neq 0$.

There are 65,536 models whose steps have weights in $\{0, 1, 2, 3\}$, and 30,307 are "interesting".

"Interesting" models

A model is interesting if it is:

- Not equivalent to a half plane model
- Not equivalent to some "interesting" model by reflection about the diagonal $x = y$
- Not equivalent to λS for some "interesting" model S, with $\lambda \neq 0$.

There are 65,536 models whose steps have weights in $\{0, 1, 2, 3\}$, and 30,307 are "interesting".

- (at least) 1457 of those are D-finite!

"Interesting" models

A model is interesting if it is:

- Not equivalent to a half plane model
- Not equivalent to some "interesting" model by reflection about the diagonal $x = y$
- Not equivalent to λS for some "interesting" model S, with $\lambda \neq 0$.

There are 65,536 models whose steps have weights in $\{0, 1, 2, 3\}$, and 30,307 are "interesting".

- (at least) 1457 of those are D-finite!
    - (at least) 79 of these are algebraic!

The Group of the Model

The Group of the Model

Let $a_{u,v}$ be the multiplicity of step (u, v) in the model M.

The Group of the Model

Let $a_{u,v}$ be the multiplicity of step (u, v) in the model M.
$K_M(x, y) = 1 - t \sum_{u,v} a_{u,v} x^u y^v$ is the *kernel polynomial*

The Group of the Model

Let $a_{u,v}$ be the multiplicity of step (u, v) in the model M.
$K_M(x, y) = 1 - t \sum_{u,v} a_{u,v} x^u y^v$ is the *kernel polynomial*

Definition (The group of the model)

The *group of the model* $G_M = \langle \Phi, \Psi \rangle$, where

$$\Phi \colon (x, y) \mapsto \left(\frac{1}{x}\frac{\sum_v a_{-1,v} y^v}{\sum_v a_{1,v} y^v},\ y\right), \Psi \colon (x, y) \mapsto \left(x,\ \frac{1}{y}\frac{\sum_u a_{u,-1} x^u}{\sum_u a_{u,1} x^u}\right)$$

The Group of the Model

Let $a_{u,v}$ be the multiplicity of step (u,v) in the model M.
$K_M(x,y) = 1 - t\sum_{u,v} a_{u,v}x^u y^v$ is the *kernel polynomial*

Definition (The group of the model)

The *group of the model* $G_M = \langle \Phi, \Psi \rangle$, where

$$\Phi\colon (x,y) \mapsto \left(\frac{1}{x}\frac{\sum_v a_{-1,v}y^v}{\sum_v a_{1,v}y^v},\ y\right), \Psi\colon (x,y) \mapsto \left(x,\ \frac{1}{y}\frac{\sum_u a_{u,-1}x^u}{\sum_u a_{u,1}x^u}\right)$$

If $g \in G_M$, $g(K(x,y)) = K(x,y)$.

The Group of the Model

Let $a_{u,v}$ be the multiplicity of step (u,v) in the model M.
$K_M(x,y) = 1 - t\sum_{u,v} a_{u,v} x^u y^v$ is the *kernel polynomial*

Definition (The group of the model)

The *group of the model* $G_M = \langle \Phi, \Psi \rangle$, where

$$\Phi\colon (x,y) \mapsto \left(\frac{1}{x}\frac{\sum_v a_{-1,v} y^v}{\sum_v a_{1,v} y^v},\ y\right), \Psi\colon (x,y) \mapsto \left(x,\ \frac{1}{y}\frac{\sum_u a_{u,-1} x^u}{\sum_u a_{u,1} x^u}\right)$$

If $g \in G_M$, $g(K(x,y)) = K(x,y)$.
Note that Φ and Ψ are involutions.

More about G_M

More about G_M

$$G_M = \big\{1, \Psi\Phi, (\Psi\Phi)^2, \ldots, (\Psi\Phi)^{n-1}, \\ \Psi, (\Psi\Phi)\Psi, (\Psi\Phi)^2\Psi, \ldots, (\Psi\Phi)^{n-1}\Psi\big\}$$

More about G_M

$$G_M = \{1, \Psi\Phi, (\Psi\Phi)^2, \ldots, (\Psi\Phi)^{n-1}, \\ \Psi, (\Psi\Phi)\Psi, (\Psi\Phi)^2\Psi, \ldots, (\Psi\Phi)^{n-1}\Psi\}$$

Since Φ, Ψ are involutions, G must be dihedral.

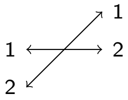
More about G_M

$$G_M = \big\{1, \Psi\Phi, (\Psi\Phi)^2, \ldots, (\Psi\Phi)^{n-1}, \\ \Psi, (\Psi\Phi)\Psi, (\Psi\Phi)^2\Psi, \ldots, (\Psi\Phi)^{n-1}\Psi\big\}$$

Since Φ, Ψ are involutions, G must be dihedral.

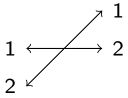
For $n \in \mathbb{N}$, D_{2n} appears if and only if $(\Psi\Phi)^n = 1$ and there is no $d|n$ such that $(\Phi\Psi)^d = 1$

Example: Gessel model with multiplicities



$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

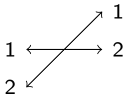
Example: Gessel model with multiplicities



$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

$$a_{1,0} = 2 = a_{-1,-1}, a_{1,1} = 1 = a_{-1,0}$$

Example: Gessel model with multiplicities

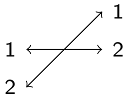


$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

$a_{1,0} = 2 = a_{-1,-1}, a_{1,1} = 1 = a_{-1,0}$

$K_{\mathcal{G}}(x,y) = 1 - t(2x + xy + x^{-1} + 2x^{-1}y^{-1})$

Example: Gessel model with multiplicities



$$\mathcal{G} = \{(1,0),(1,0),(1,1),(-1,0),(-1,-1),(-1,-1)\}$$

$$a_{1,0} = 2 = a_{-1,-1}, a_{1,1} = 1 = a_{-1,0}$$
$$K_{\mathcal{G}}(x,y) = 1 - t(2x + xy + x^{-1} + 2x^{-1}y^{-1})$$
$$G_{\mathcal{G}} = \left\langle \left(\frac{1}{xy}, y\right), \left(x, \frac{2}{x^2 y}\right) \right\rangle \cong D_8$$

Method

Method

Idea: fix a group and find out which models have this group.

Method

Idea: fix a group and find out which models have this group.

For fixed n, we can explicitly write down $(\Phi\Psi)^n(x,y) = \left(\frac{p}{q}, \frac{r}{s}\right)$, where p, q, r, s are polynomials in the variables $a_{-1,-1}, \ldots, a_{1,1}$ and x, y.

Method

Idea: fix a group and find out which models have this group.

For fixed n, we can explicitly write down $(\Phi\Psi)^n(x,y) = \left(\frac{p}{q}, \frac{r}{s}\right)$, where p, q, r, s are polynomials in the variables $a_{-1,-1}, \ldots, a_{1,1}$ and x, y.

$(\Phi\Psi)^n(x,y) = (x,y)$. Compare coefficients with respect to x, y.

Method

Idea: fix a group and find out which models have this group.

For fixed n, we can explicitly write down $(\Phi\Psi)^n(x,y) = \left(\frac{p}{q}, \frac{r}{s}\right)$, where p, q, r, s are polynomials in the variables $a_{-1,-1}, \ldots, a_{1,1}$ and x, y.

$(\Phi\Psi)^n(x,y) = (x,y)$. Compare coefficients with respect to x, y.

Obtain a system S of nonlinear equations for the $a_{u,v}$. The points $(a_{-1,-1}, \ldots, a_{1,1}) \in \mathbb{C}^8$ satisfying this system form an algebraic variety V.

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Determine the variety for a fixed n:

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Determine the variety for a fixed n:

> Find generators for the irreducible components of the radical $\sqrt{I(V)}$.

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Determine the variety for a fixed n:

Find generators for the irreducible components of the radical $\sqrt{I(V)}$.

Use the generators to find relations among the $a_{u,v}$

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Determine the variety for a fixed n:

- Find generators for the irreducible components of the radical $\sqrt{I(V)}$.
- Use the generators to find relations among the $a_{u,v}$

Prove the resulting generating functions are D-finite:

Task: Determine this variety for fixed n. Then prove that the resulting generating functions are D-finite.

Determine the variety for a fixed n:

- Find generators for the irreducible components of the radical $\sqrt{I(V)}$.
- Use the generators to find relations among the $a_{u,v}$

Prove the resulting generating functions are D-finite:

- Use the kernel, orbit sum, or half orbit sum method to prove D-finiteness of the generating function $F_M(x, y, z, t)$ for every M in the family.

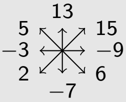
D_4

D_4

Family 0

Defining equations:

$a_{0,1}a_{1,-1} = a_{0,-1}a_{1,1}$,
$a_{-1,1}a_{1,-1} = a_{-1,-1}a_{1,1}$,
$a_{-1,1}a_{0,-1} = a_{-1,-1}a_{0,1}$

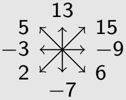


D_4

Family 0

Defining equations:

$a_{0,1}a_{1,-1} = a_{0,-1}a_{1,1}$,
$a_{-1,1}a_{1,-1} = a_{-1,-1}a_{1,1}$,
$a_{-1,1}a_{0,-1} = a_{-1,-1}a_{0,1}$



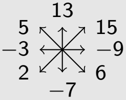
All models in Family 0 have a D-finite generating function.

D_4

Family 0

Defining equations:

$a_{0,1}a_{1,-1} = a_{0,-1}a_{1,1},$
$a_{-1,1}a_{1,-1} = a_{-1,-1}a_{1,1},$
$a_{-1,1}a_{0,-1} = a_{-1,-1}a_{0,1}$



All models in Family 0 have a D-finite generating function.

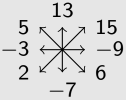
All models with group D4 belong to Family 0.

D_4

Family 0

Defining equations:

$a_{0,1}a_{1,-1} = a_{0,-1}a_{1,1}$,
$a_{-1,1}a_{1,-1} = a_{-1,-1}a_{1,1}$,
$a_{-1,1}a_{0,-1} = a_{-1,-1}a_{0,1}$



All models in Family 0 have a D-finite generating function.

All models with group D4 belong to Family 0.

Note: $a_{0,1}a_{1,-1}$ (and other products) need not be integers, or even rational numbers.

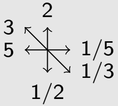
D_6

Family 1a

Defining equations:

$a_{1,1} = a_{-1,-1} = 0,$

$a_{-1,1}a_{1,-1} = a_{-1,0}a_{1,0} = a_{0,1}a_{0,-1}$

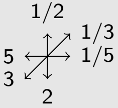


Family 1b

Defining equations:

$a_{1,-1} = a_{-1,1} = 0,$

$a_{-1,0}a_{1,0} = a_{-1,-1}a_{1,1} = a_{0,-1}a_{0,1}$



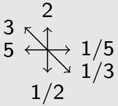
D_6

Family 1a

Defining equations:

$a_{1,1} = a_{-1,-1} = 0,$

$a_{-1,1}a_{1,-1} = a_{-1,0}a_{1,0} = a_{0,1}a_{0,-1}$

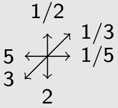


Family 1b

Defining equations:

$a_{1,-1} = a_{-1,1} = 0,$

$a_{-1,0}a_{1,0} = a_{-1,-1}a_{1,1} = a_{0,-1}a_{0,1}$



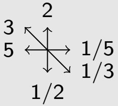
Both families have D-finite generating functions.

D_6

Family 1a

Defining equations:

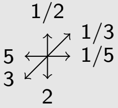
$$a_{1,1} = a_{-1,-1} = 0,$$
$$a_{-1,1}a_{1,-1} = a_{-1,0}a_{1,0} = a_{0,1}a_{0,-1}$$



Family 1b

Defining equations:

$$a_{1,-1} = a_{-1,1} = 0,$$
$$a_{-1,0}a_{1,0} = a_{-1,-1}a_{1,1} = a_{0,-1}a_{0,1}$$



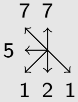
Both families have D-finite generating functions.
Family 1b models actually have *algebraic* GFs!

D_6 (cont)

Family 2a

Defining equations:

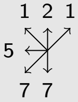
$a_{1,0} = a_{1,1} = 0$, $a_{0,-1}a_{-1,1} = 2a_{0,1}a_{-1,-1}$,
$a_{0,-1}^2 = 4a_{1,-1}a_{-1,-1}$, $a_{0,-1}a_{0,1} = 2a_{-1,1}a_{1,-1}$



Family 2b

Defining equations:

$a_{1,0} = a_{1,-1} = 0$, $a_{0,1}a_{-1,-1} = 2a_{0,-1}a_{-1,1}$,
$a_{0,1}^2 = 4a_{1,1}a_{-1,1}$, $a_{0,1}a_{0,-1} = 2a_{-1,-1}a_{1,1}$

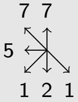


D_6 (cont)

Family 2a

Defining equations:

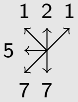
$a_{1,0} = a_{1,1} = 0$, $a_{0,-1}a_{-1,1} = 2a_{0,1}a_{-1,-1}$,
$a_{0,-1}^2 = 4a_{1,-1}a_{-1,-1}$, $a_{0,-1}a_{0,1} = 2a_{-1,1}a_{1,-1}$



Family 2b

Defining equations:

$a_{1,0} = a_{1,-1} = 0$, $a_{0,1}a_{-1,-1} = 2a_{0,-1}a_{-1,1}$,
$a_{0,1}^2 = 4a_{1,1}a_{-1,1}$, $a_{0,1}a_{0,-1} = 2a_{-1,-1}a_{1,1}$



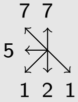
Both families have D-finite generating functions.

D_6 (cont)

Family 2a

Defining equations:

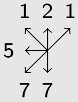
$a_{1,0} = a_{1,1} = 0$, $a_{0,-1}a_{-1,1} = 2a_{0,1}a_{-1,-1}$,
$a_{0,-1}^2 = 4a_{1,-1}a_{-1,-1}$, $a_{0,-1}a_{0,1} = 2a_{-1,1}a_{1,-1}$



Family 2b

Defining equations:

$a_{1,0} = a_{1,-1} = 0$, $a_{0,1}a_{-1,-1} = 2a_{0,-1}a_{-1,1}$,
$a_{0,1}^2 = 4a_{1,1}a_{-1,1}$, $a_{0,1}a_{0,-1} = 2a_{-1,-1}a_{1,1}$



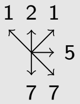
Both families have D-finite generating functions.
Family 2b models have *algebraic* GFs.

D_6 (still cont)

Family 3a

Defining equations:

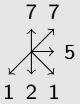
$$a_{-1,0} = a_{-1,-1} = 0,\ a_{0,1}a_{1,-1} = 2a_{0,-1}a_{1,1},$$
$$a_{0,1}^2 = 4a_{-1,1}a_{1,1},\ a_{0,1}a_{0,-1} = 2a_{1,-1}a_{-1,1}$$



Family 3b

Defining equations:

$$a_{-1,0} = a_{-1,1} = 0,\ a_{0,-1}a_{1,1} = 2a_{0,1}a_{1,-1},$$
$$a_{0,-1}^2 = 4a_{-1,-1}a_{1,-1},\ a_{0,-1}a_{0,1} = 2a_{1,1}a_{-1,-1}$$

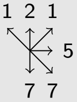


D_6 (still cont)

Family 3a

Defining equations:

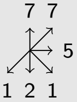
$a_{-1,0} = a_{-1,-1} = 0$, $a_{0,1}a_{1,-1} = 2a_{0,-1}a_{1,1}$,
$a_{0,1}^2 = 4a_{-1,1}a_{1,1}$, $a_{0,1}a_{0,-1} = 2a_{1,-1}a_{-1,1}$



Family 3b

Defining equations:

$a_{-1,0} = a_{-1,1} = 0$, $a_{0,-1}a_{1,1} = 2a_{0,1}a_{1,-1}$,
$a_{0,-1}^2 = 4a_{-1,-1}a_{1,-1}$, $a_{0,-1}a_{0,1} = 2a_{1,1}a_{-1,-1}$



Again, all families have D-finite generating functions.

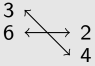
D_8

Family 4a

Defining equations:

$a_{1,-1}a_{-1,1} = a_{1,0}a_{-1,0}$,

$a_{1,1} = a_{0,1} = a_{0,-1} = a_{-1,-1} = 0$

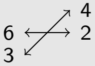


Family 4b

Defining equations:

$a_{1,1}a_{-1,-1} = a_{1,0}a_{-1,0}$,

$a_{1,-1} = a_{0,1} = a_{0,-1} = a_{-1,1} = 0$



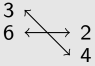
D_8

Family 4a

Defining equations:

$a_{1,-1}a_{-1,1} = a_{1,0}a_{-1,0}$,

$a_{1,1} = a_{0,1} = a_{0,-1} = a_{-1,-1} = 0$

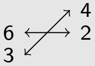


Family 4b

Defining equations:

$a_{1,1}a_{-1,-1} = a_{1,0}a_{-1,0}$,

$a_{1,-1} = a_{0,1} = a_{0,-1} = a_{-1,1} = 0$



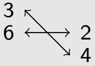
Both families have D-finite generating functions.

D_8

Family 4a

Defining equations:

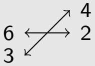
$$a_{1,-1}a_{-1,1} = a_{1,0}a_{-1,0},$$
$$a_{1,1} = a_{0,1} = a_{0,-1} = a_{-1,-1} = 0$$



Family 4b

Defining equations:

$$a_{1,1}a_{-1,-1} = a_{1,0}a_{-1,0},$$
$$a_{1,-1} = a_{0,1} = a_{0,-1} = a_{-1,1} = 0$$



Both families have D-finite generating functions.

Family 4b models have *algebraic* GFs

D_{10}

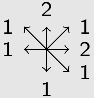
D_{10}

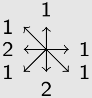
We can find families as in the other cases, but are not sure we have the full characterization.

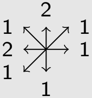
D_{10}

We can find families as in the other cases, but are not sure we have the full characterization.

3 models with group D_{10}



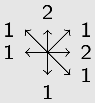


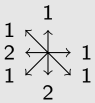


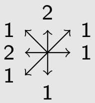
D_{10}

We can find families as in the other cases, but are not sure we have the full characterization.

3 models with group D_{10}





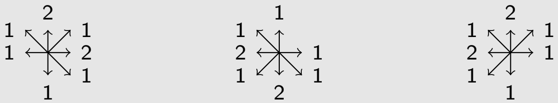


These models do not fit into any of the previous families.

D_{10}

We can find families as in the other cases, but are not sure we have the full characterization.

3 models with group D_{10}



These models do not fit into any of the previous families.

Their GFs are (probably*) algebraic.

What next?

What next?

Finite group $\iff$ D-finite generating function?

What next?

Finite group $\iff$ D-finite generating function?

This is true in the multiplicity 1 case. It seems that it is true here as well. Is there a combinatorial proof?

What next?

Finite group $\iff$ D-finite generating function?

This is true in the multiplicity 1 case. It seems that it is true here as well. Is there a combinatorial proof?

Are there models whose groups are larger than D_{10}?

What next?

Finite group $\iff$ D-finite generating function?

This is true in the multiplicity 1 case. It seems that it is true here as well. Is there a combinatorial proof?

Are there models whose groups are larger than D_{10}?

We've done calculations for multiplicity 4 and 5 and found nothing bigger.

What next?

Finite group $\iff$ D-finite generating function?

This is true in the multiplicity 1 case. It seems that it is true here as well. Is there a combinatorial proof?

Are there models whose groups are larger than D_{10}?

We've done calculations for multiplicity 4 and 5 and found nothing bigger.

Is it possible to do a similar classification in the 3d case?

What next?

Finite group $\iff$ D-finite generating function?

> This is true in the multiplicity 1 case. It seems that it is true here as well. Is there a combinatorial proof?

Are there models whose groups are larger than D_{10}?

> We've done calculations for multiplicity 4 and 5 and found nothing bigger.

Is it possible to do a similar classification in the 3d case?

Prove that the remaining 28,850 "interesting" cases have non-D-finite GFs.