Subwords and Plane Partitions

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IMA and LaCIM

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Theorem

Theorem (H.,W. (2015))

There is a bijection between centrally symmetric k-triangulations of a $2(n+k)$-gon and plane partitions in a $n \times n \times k$ box.

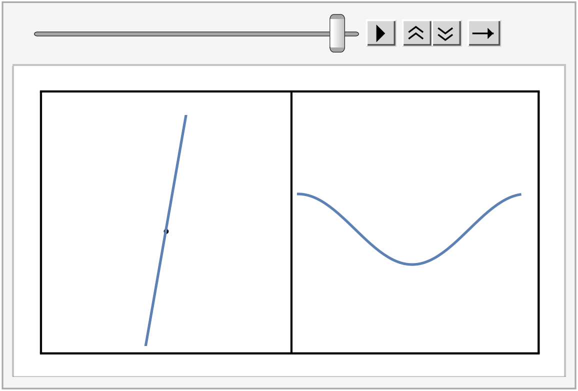
Outline

Points and Pseudolines (V. Pilaud)

There is a duality between points in $\mathbb{R}^2$ and pseudolines.

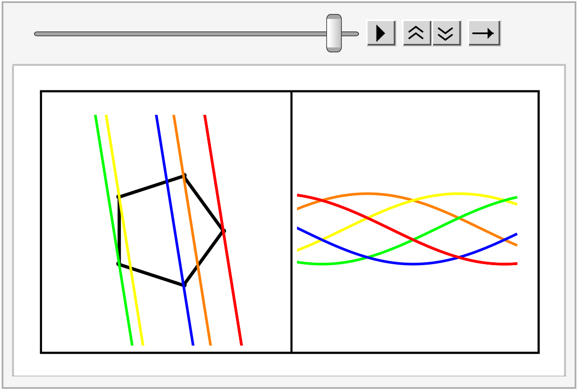
A point (x, y) is mapped to the curve (θ, d), where

- θ is the angle between a line through (x, y) and the x-axis, and
- $d := -x \sin(\theta) + y \cos(\theta)$.



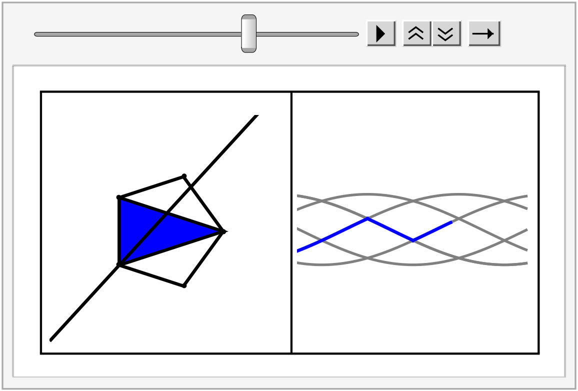
Points and Pseudoline Arrangements

There is a duality between vertices of a regular $(n+2)$-gon and a collection of $n+2$ pseudolines.



Points and Pseudoline Arrangements

There is a duality between triangulations of a regular $(n+2)$-gon and certain connected pieces of the pseudoline arrangement.
A triangle T is mapped to those pieces of pseudolines that come from angles defining lines that pass through the interior of T.



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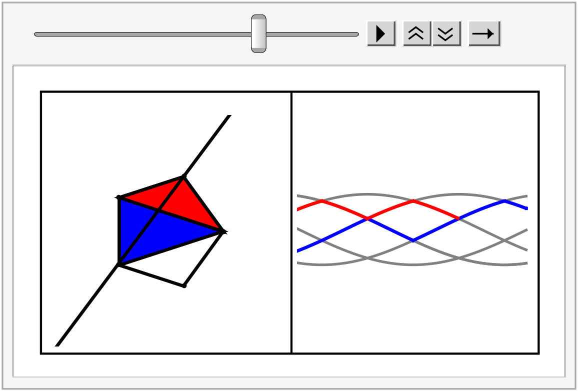

Points and Pseudoline Arrangements

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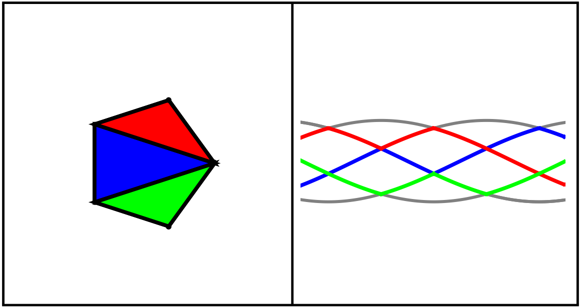
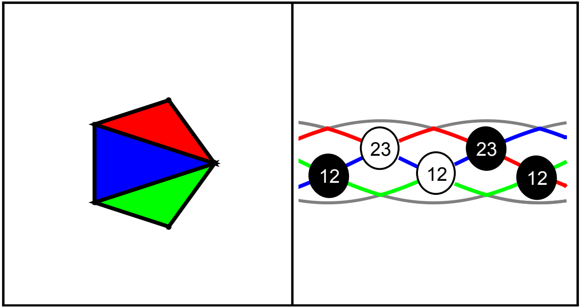


Characterization

A triangle T is mapped to those strands coming from angles corresponding to lines that pass through the interior of T.

- The top-most and bottom-most pieces of the pseudolines cannot occur as a piece of pseudoline coming from a triangulation.
- The rest of the arrangement is covered.
- Every pseudoline crosses every other pseudoline exactly once.



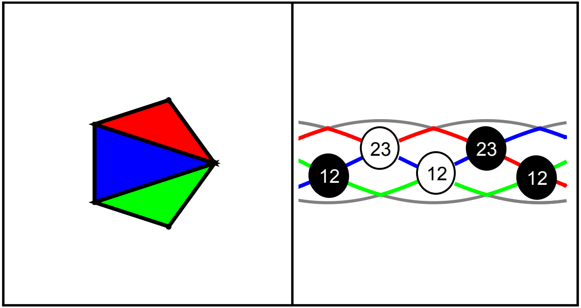
Characterization

Theorem (V. Pilaud, M. Pocchiola (2012))

The triangulations of a regular $(n+2)$-gon are in bijection with subwords of the word

$$(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2})(s_1 s_2 \cdots s_{n-3}) \cdots (s_1 s_2)(s_1).$$

that are reduced words for the longest element $w_\circ \in \mathfrak{S}_n$

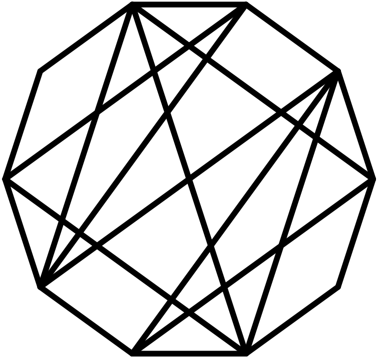


Definition of Multitriangulations

Definition (V. Capoyleas and J. Pach (1992))

A *k-triangulation* T of a regular convex n-gon is a maximal set of diagonals of the n-gon such that no $k+1$ of them are mutually crossing.

The role of triangles in the duality to pseudoline arrangements is now played by k-stars.



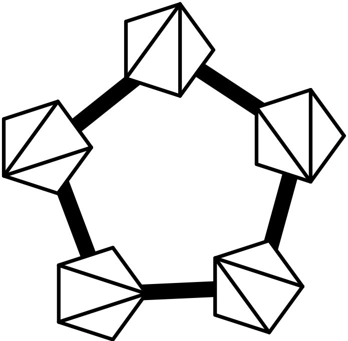
Characterization

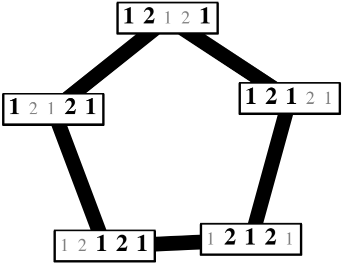
Theorem (V. Pilaud, M. Pocchiola (2012))

The k-triangulations of an $(n+2k)$-gon are in bijection with subwords of the word

$$(s_1 s_2 \cdots s_{n-1})^k (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2})(s_1 s_2 \cdots s_{n-3}) \cdots (s_1 s_2)(s_1).$$

that are reduced words for the longest element $w_\circ \in \mathfrak{S}_n$.





Generalization

Let W be a Coxeter group, c a Coxeter element, $w(c)$ the c-sorting word for w.

Definition (C. Ceballos, J. P. Labbé, C. Stump (2011))

The k-triangulations for W are the subwords of the word

$$c^k w_\circ(c)$$

that are reduced words for the longest element $w_\circ \in W$.

Enumeration

Theorem (J. Jonsson (2005))

There are

$$\prod_{1 \leq i \leq j \leq n-1} \frac{i+j+2k}{i+j}$$

k-triangulations of an $(n+2k)$-gon.

Problem

Is there a bijection between k-triangulations of an $(n+2k)$-gon and plane partitions of height k in a staircase $(n-1, n-2, \ldots, 1)$?

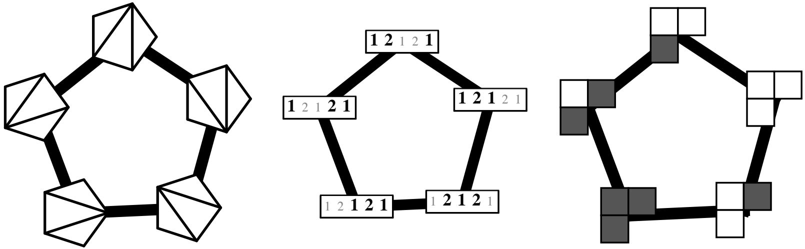
Bijections

Theorem (S. Elizalde (2006))

There is a bijection between 2-triangulations of an $(n+4)$-gon and plane partitions of height 2 in a staircase $(n-1, n-2, \ldots, 1)$.

Theorem (L. Serrano, C. Stump (2012))

There is a bijection between k-triangulations of an $(n+2k)$-gon and plane partitions of height k in a staircase $(n-1, n-2, \ldots, 1)$.



Edelman-Greene

Theorem (R. Stanley (1984), P. Edelman and C. Greene (1987))

There is a bijection between linear extensions of the staircase $(n-1, n-2, \ldots, 1)$ *and reduced words for the longest element* $w_\circ \in \mathfrak{S}_n$.

$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

3	1

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$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

1	1
3	2

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$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

1	2
3	

1	3
2	

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There is a bijection between linear extensions of the staircase $(n-1, n-2, \ldots, 1)$ *and reduced words for the longest element* $w_\circ \in \mathfrak{S}_n$.

$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

1	2	3
3		

1	3	4
2		

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$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

1	2	3
2		
3		

1	3	4
2		
5		

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$$w_\circ = s_3 s_1 s_2 s_3 s_1 s_2$$

1	2	3
2	3	
3		

1	3	4
2	6	
5		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	$\textcircled{s_3}$	$\textcircled{s_1}$	$\textcircled{s_2}$	$\textcircled{s_3}$	s_1	s_2	s_3	$\textcircled{s_1}$	$\textcircled{s_2}$	s_1

1	3	4
2	6	
5		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1

1	3	4
2	6	
5		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	$\textcircled{$s_3$}$	$\textcircled{$s_1$}$	$\textcircled{$s_2$}$	$\textcircled{$s_3$}$	s_1	s_2	s_3	$\textcircled{$s_1$}$	$\textcircled{$s_2$}$	s_1

1	2	4
2	6	
5		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	$\textcircled{$s_3$}$	$\textcircled{$s_1$}$	$\textcircled{$s_2$}$	$\textcircled{$s_3$}$	s_1	s_2	s_3	$\textcircled{$s_1$}$	$\textcircled{$s_2$}$	s_1

1	2	2
2	6	
5		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	$\textcircled{s_3}$	$\textcircled{s_1}$	$\textcircled{s_2}$	$\textcircled{s_3}$	s_1	s_2	s_3	$\textcircled{s_1}$	$\textcircled{s_2}$	s_1

1	2	2
2	6	
4		

Lifting Edelman-Greene

1	1	1	2	2	2	3	3	3	4	4	5
s_1	s_2	$\textcircled{s_3}$	$\textcircled{s_1}$	$\textcircled{s_2}$	$\textcircled{s_3}$	s_1	s_2	s_3	$\textcircled{s_1}$	$\textcircled{s_2}$	s_1

1	2	2
2	4	
4		

Lifting Edelman-Greene

1	1	1	|	2	2	2	|	3	3	3	|	4	4	|	5
s_1	s_2	$\enclose{circle}{s_3}$	|	$\enclose{circle}{s_1}$	$\enclose{circle}{s_2}$	$\enclose{circle}{s_3}$	|	s_1	s_2	s_3	|	$\enclose{circle}{s_1}$	$\enclose{circle}{s_2}$	|	s_1

0	1	1
0	2	
1		

Lifting Edelman-Greene

- S. Billey, W. Jockush, and R. Stanley - Combinatorial Properties of Schubert Polynomials (1993),
- F. Fomin, A. Kirillov - Reduced Words and Plane Partitions (1997),
- A. Woo - Catalan Numbers and Schubert Polynomials (2004),
- J. Morse, A. Schilling - Crystal operators and flag Gromov-Witten invariants (2014).

Why?

Theorem (J. Morse, A. Schilling (2014))

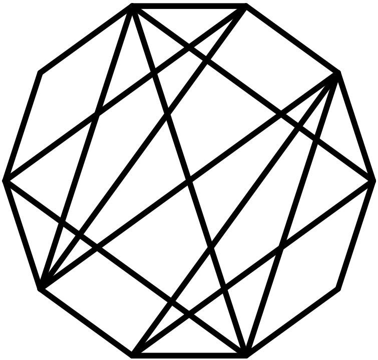
The subwords of $(s_1 s_2 \cdots s_{n-1})^{n+k-1}$ *that are reduced words for the longest element* $w_\circ \in \mathfrak{S}_n$ *may be given a crystal structure of type* A_{n+k-2} *isomorphic to* $B(\omega_1 + \cdots + \omega_{n-1})$.

- k-triangulations embed in this model,
- which is also indexed by semistandard tableaux.

Definition of Centrally-Symmetric Multitriangulations

Definition (D. Soll and V Welker (2009))

A centrally symmetric k-triangulation of a $2n$-gon is a k-triangulation that is invariant under rotation of the $2n$-gon by π radians.



Enumeration

Theorem (M. Rubey and C. Stump (2009); conjectured by D. Soll and V Welker)

There are

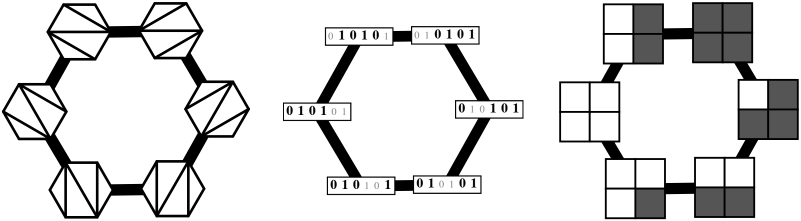
$$\prod_{h=1}^{k}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{h+i+j-1}{h+i+j-2}$$

centrally-symmetric k-triangulations of a $2(n+k)$-gon.

Bijection

Theorem (H., W. (2015))

There is a bijection between centrally symmetric k-triangulations of an $2(n+k)$-gon and plane partitions in an $n \times n \times k$ box.



Characterization

Theorem (C. Ceballos, J. P. Labbé, C. Stump (2011))

The centrally-symmetric k-triangulations of an $2(n+k)$-gon are in bijection with the subwords of the word

$$(s_1 s_2 \cdots s_n)^{k+n}$$

that are reduced words for the longest element $w_\circ \in B_n$.

Kraśkiewicz Insertion

Theorem (W. Kraśkiewicz (1989), M. Haiman (1992))

There is a bijection between linear extensions of the shifted trapezoid $(2n+1, 2n-1, \ldots, 3, 1)$ *and reduced words for the longest element* $w_\circ \in B_n$.

$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

0	1

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0	1

1	2

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$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\bar{1}$	0
	0

1	2
	3

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$\bar{1}$	0	1
	0	

1	2	4
	3	

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$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\bar{1}$	0	1	2
	0		

1	2	4	5
	3		

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$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\overline{2}$	0	1	2
	0	1	

1	2	4	5
	3	6	

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$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\overline{2}$	$\overline{1}$	0	2
	$\overline{1}$	0	
		0	

1	2	4	5
	3	6	
		7	

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$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\bar{2}$	$\bar{1}$	0	1
	$\bar{1}$	0	1
		0	

1	2	4	5
	3	6	8
		7	

Kraśkiewicz Insertion

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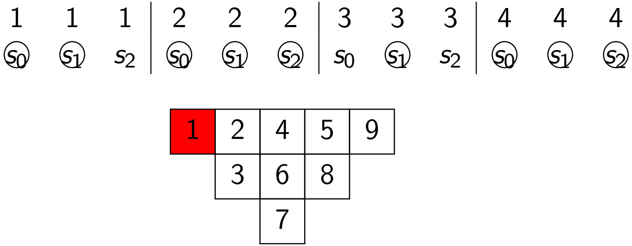
$$w_\circ = s_0 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_2$$

$\bar{2}$	$\bar{1}$	0	1	2
	$\bar{1}$	0	1	
		0		

1	2	4	5	9
	3	6	8	
		7		

Lifting Kraśkiewicz



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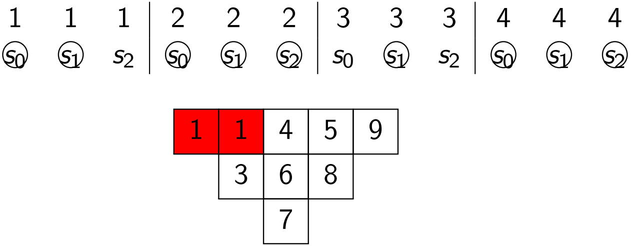

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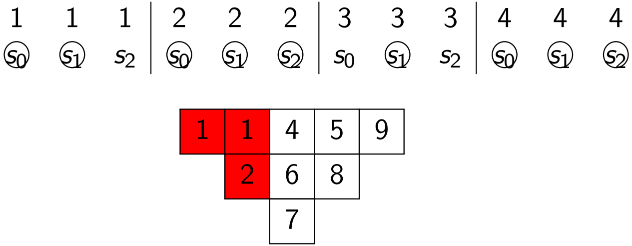

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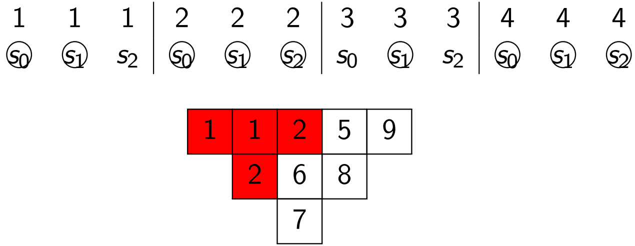

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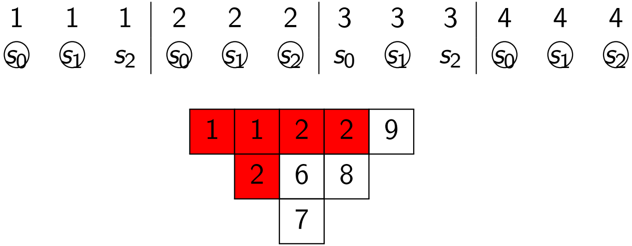

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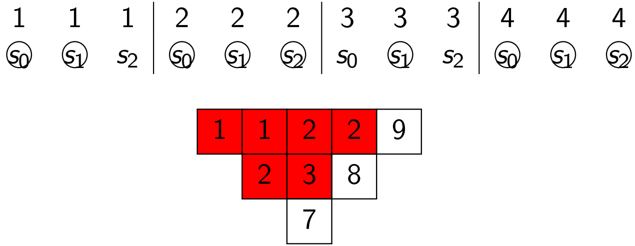

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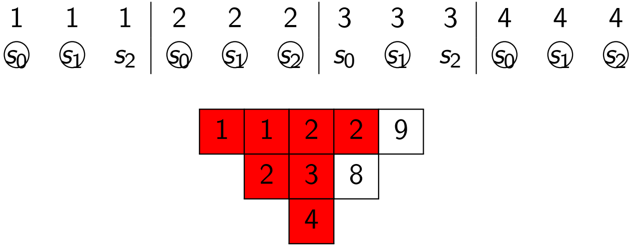
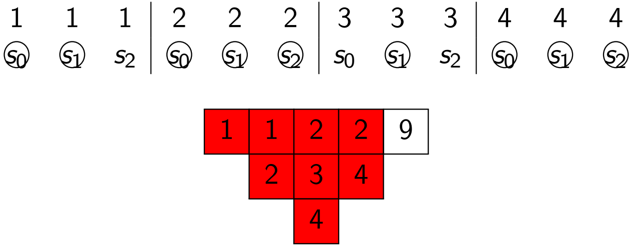
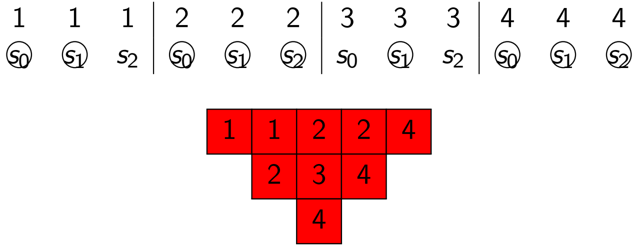


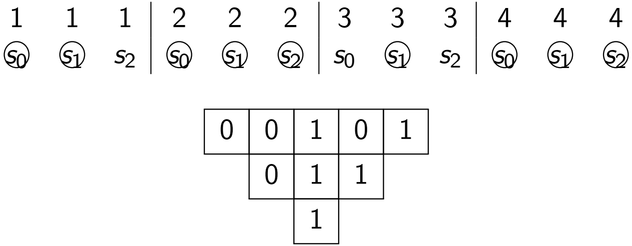
Lifting Kraśkiewicz



Lifting Kraśkiewicz



Lifting Kraśkiewicz



Rectification

Theorem (M. Haiman (1992))

There is a bijection between linear extensions of a square and linear extensions of a shifted trapezoid.

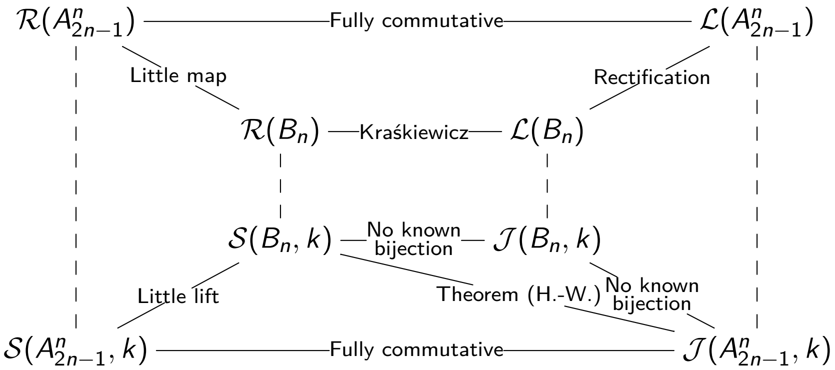
1	2	4	5	9
	3	6	8	
		7		

1	2	5
3	4	6
7	8	9

0	0	1	0	1
	0	1	1	
		1		

0	0	1
0	0	1
1	1	1

Proof

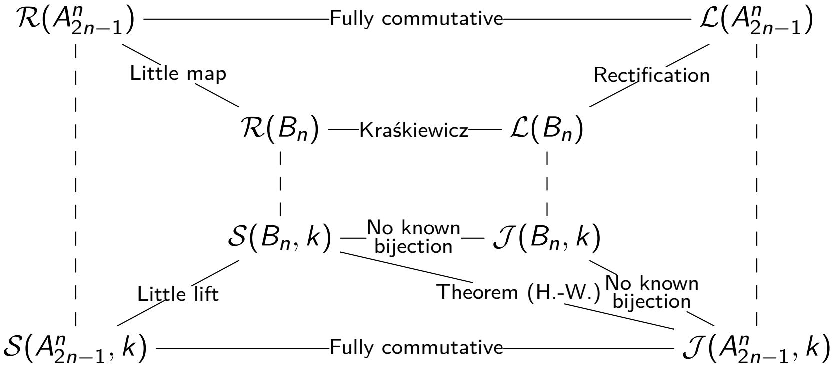


$\mathcal{R}(A_{2n-1}^n)$
Fully commutative
$\mathcal{L}(A_{2n-1}^n)$
Little map
Rectification
$\mathcal{R}(B_n)$
Kraśkiewicz
$\mathcal{L}(B_n)$
$\mathcal{S}(B_n, k)$
No known bijection
$\mathcal{J}(B_n, k)$
Little lift
Theorem (H.-W.)
No known bijection
$\mathcal{S}(A_{2n-1}^n, k)$
Fully commutative
$\mathcal{J}(A_{2n-1}^n, k)$

Future Directions

- Type B/C crystal structure on subwords?
- Lift the correspondence between reduced words and linear extensions in H_3?

Thank You



$\mathcal{R}(A^n_{2n-1})$
Fully commutative
$\mathcal{L}(A^n_{2n-1})$
Little map
Rectification
$\mathcal{R}(B_n)$
Kraśkiewicz
$\mathcal{L}(B_n)$
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Theorem (H.-W.)
No known bijection
$\mathcal{S}(A^n_{2n-1}, k)$
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