Coxeter-biCatalan Combinatorics

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Joint with Nathan Reading

Coxeter-Catalan Combinatorics

For each finite root system Φ with Weyl group W and Coxeter element c, the following objects are in bijection:

- Clusters in the associated finite type cluster algebra
- Vertices in the generalized associahedron
- Maximal cones in the c-Cambrian fan
- Elements of the c-Cambrian lattice (c-sortable elements)
- Nonnesting partitions: Antichains in the root poset
- Elements of the noncrossing partition lattice $NC_c(W)$

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Coxeter-biCatalan Combinatorics

A variation of Catalan-Combinatorics. Now we're counting:

- Maximal cones in a certain refinement of the c-Cambrian fan
- An analogue of the Cambrian lattice and sortable elements
- An analogue of non-nesting partitions

Motivation and Definition

Definition (BiCambrian Fan)

We define the **biCambrian fan biCamb**(W, c) to be the coarsest common refinement of the c-Cambrian fan and the c^{-1}-Cambrian fan.

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- Motivation: the linear Coxeter elements and Baxter numbers
  - Maximal cones of $bC(A_n, c)$ for c linear are in bijection with Baxter permutations (and diagonal rectangulations, twin binary trees, etc.)
  - Dilks generalized to type B
  - Independent work by Chatel and Pilaud on Twin-Cambrian Trees

Pleasant Surprises

Theorem

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Conjecture

For all crystallographic finite root systems, (W, Φ)*, the bipartitate biCambrian fan* $\mathbf{biCamb}(W, c)$ *is a simplicial fan.*

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For all crystallographic finite root systems, (W, Φ), the bipartitate biCambrian fan $\mathbf{biCamb}(W, c)$ is a simplicial fan.

Theorem

If W is of type A or B and c is a bipartite Coxeter element, then $\mathbf{biCamb}(W, c)$ is a simplicial fan.

We have verified the conjecture by computer up to rank 6.

Main Results Cont.

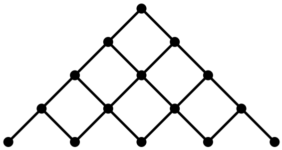
Theorem

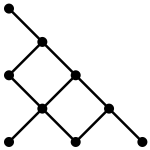
- *The bipartite biCambrian fan* $\mathbf{biCamb}(A_n, c)$ *has* $\binom{2n}{n}$ *maximal cones and h-polynomial* $\sum_{k=0}^{n} \binom{n}{k}^2 x^k$.
- *The bipartite biCambrian fan* $\mathbf{biCamb}(B_n, c)$ *has* 2^{2n-1} *maximal cones and h-polynomial* $\sum_{k=0}^{n} \binom{2n}{2k} x^k$.
- *The bipartite biCambrian fan* $\mathbf{biCamb}(D_n, c)$ *has* $6 \cdot 4^{n-2} - 2\binom{2n-4}{n-2}$ *maximal cones.*

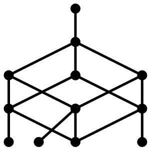
Doubled Root Poset

Definition

The **doubled root poset** consists of the root poset, together with a disjoint dual copy of the root poset, identified on the simple roots.







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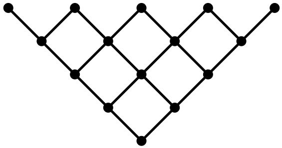

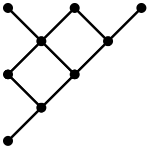

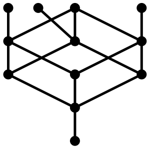

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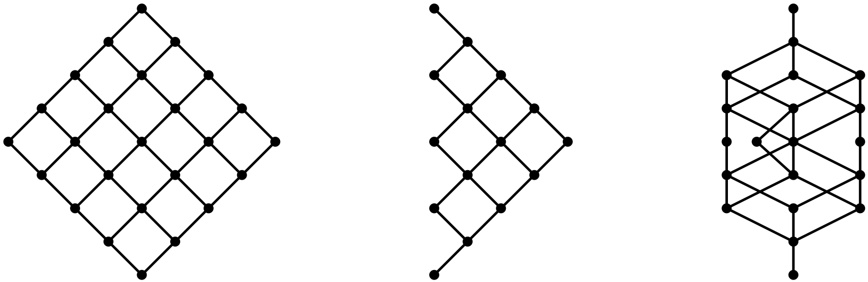

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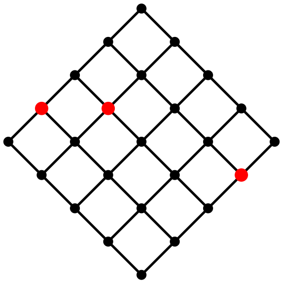

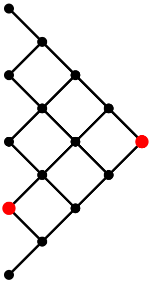

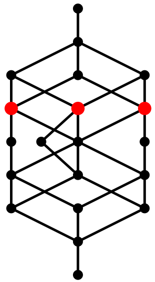

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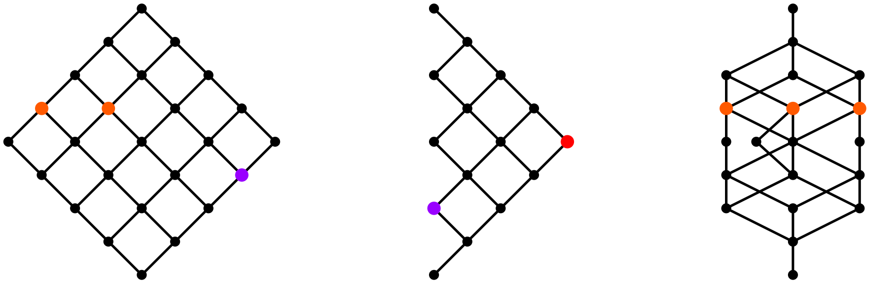

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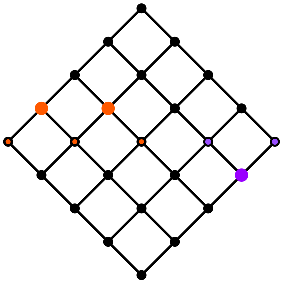

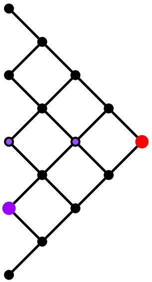

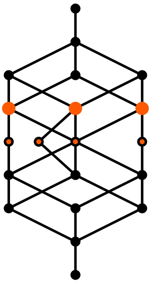

Counting Antichains in the Doubled Root Poset

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3. α is in the support of a non-simple root in A that is in the 'top' of the doubled root poset
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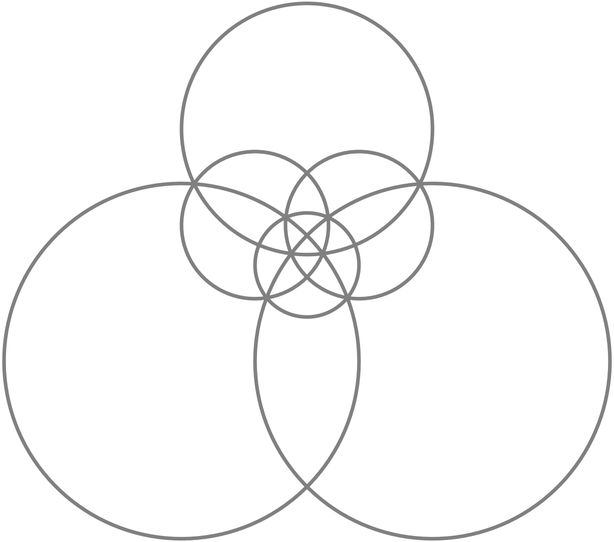
Example A_3



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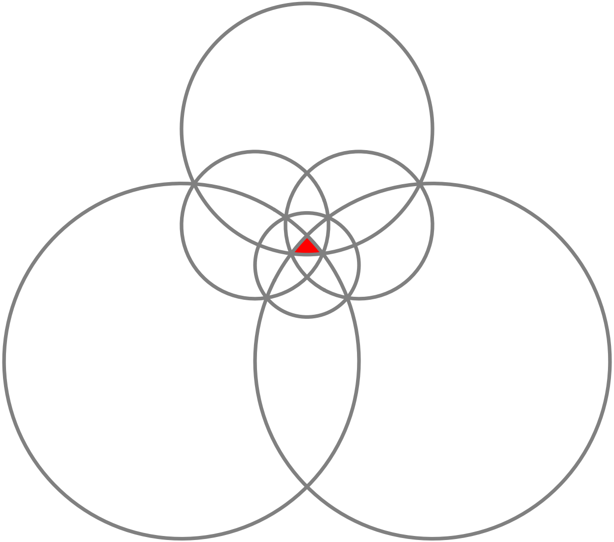

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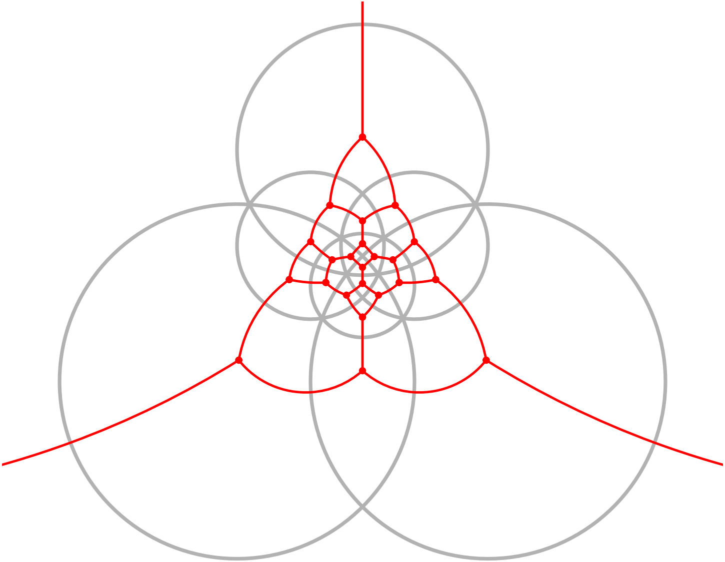

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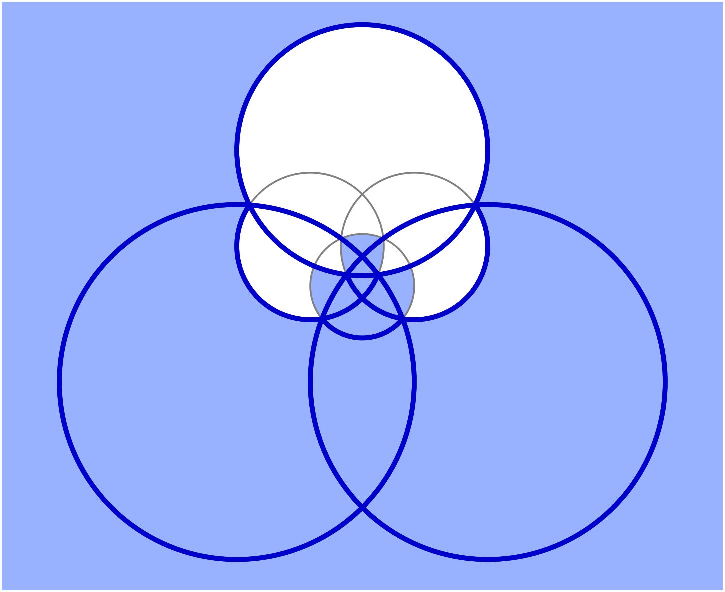

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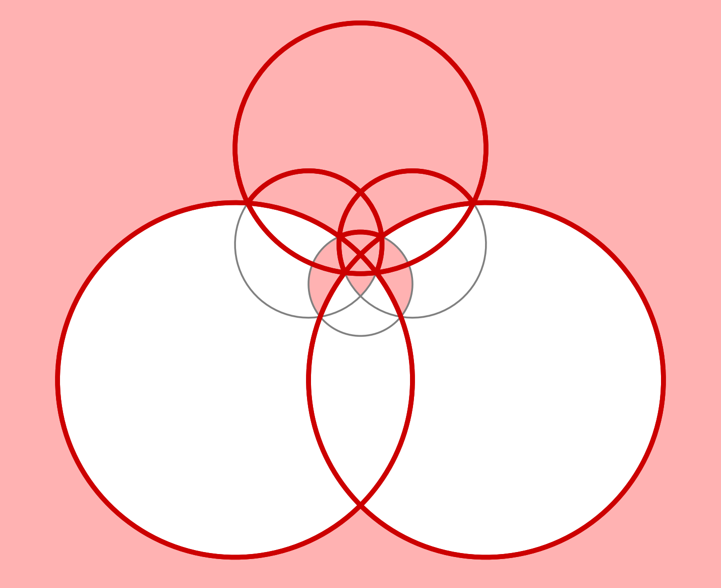

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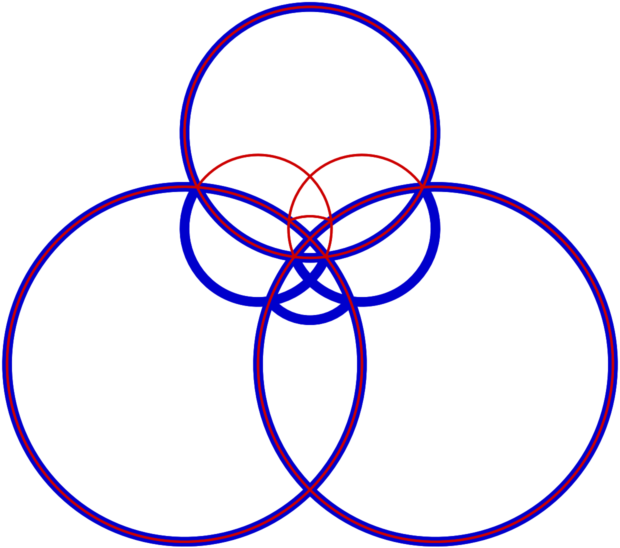

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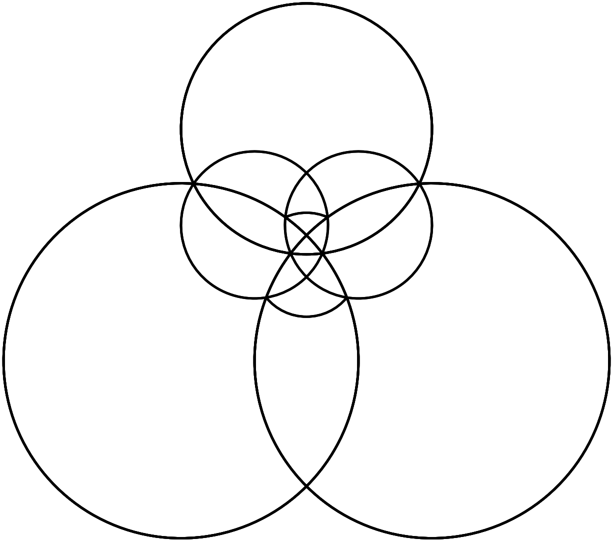

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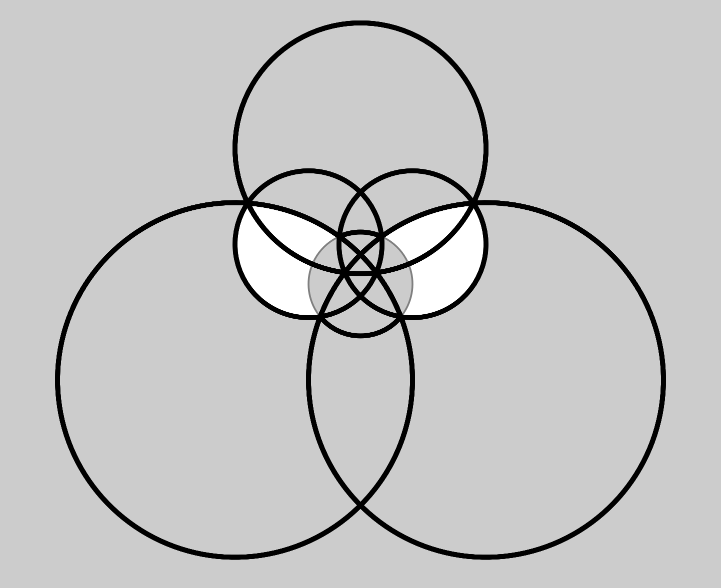

Counting biSortable Elements in type A

Take-away

We're really counting certain elements of the Coxeter group.

Definition

c-**bisortable** elements in W are the bottom elements of the maximal cones in the biCambrian Fan.

There is a bijection between non-crossing arc diagrams and permutations that restricts to a bijection from bipartite bisortable elements to **alternating arc diagrams**.

Alternating Arc Diagrams Examples

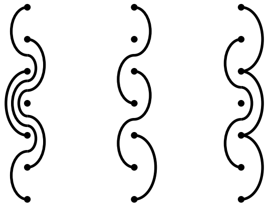

Figure: Some alternating noncrossing arc diagrams

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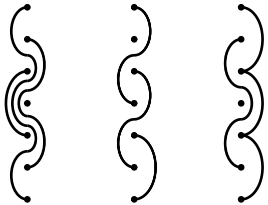

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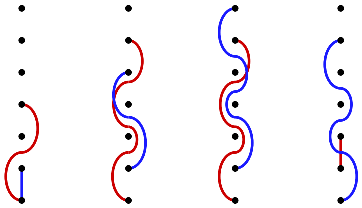

Figure: Non-diagrams

Counting Alternating Arc Diagrams

The arcs in a diagram represent the lower walls of some maximal cone in the biCambrian fan. We want to prove:

Theorem

There are $\binom{n}{k}^2$ alternating arc diagrams on $n+1$ vertices with k arcs.

- Goal: For a pair of k-subsets (S, T) of $[n]$, construct an alternating arc diagram with k arcs.

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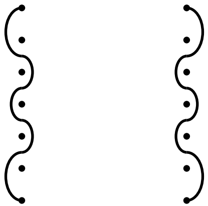
- Start with diagrams that have a single arc:



Inductive Step

1. Inductive step: Break diagrams wherever they don't overlap. Apply the map to single arc diagrams.

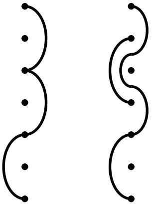


Figure: Break left diagram into 3 pieces. Break right diagram into 2 pieces.

2. For each overlapping diagram, read off top and bottom end points.

Open Questions

- Proof that the bipartite biCambrian fan is simplicial in general
- A combinatorial description of bisortable elements
- A description of the generators of the bipartite biCambrian congruence
- Enumeration of bisortable elements that is independent from enumeration of sortable elements

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