Characters, Derangements and Descents for the Hyperoctahedral Group

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July 9, 2015

Outline

1. Motivation
2. Character formulas and descents for the symmetric group
3. Character formulas and descents for the hyperoctahedral group
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Much of the motivation comes from certain equidistribution results in enumerative combinatorics. We let

- $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$,
- $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$

and for $w \in \mathfrak{S}_n$

- $\mathrm{inv}(w) \ := \ \# \{1 \le i < j \le n : \ w(i) > w(j)\}$ be the number of inversions of w,
- $\mathrm{maj}(w) \ := \ \sum_{j \in \mathrm{Des}(w)} j$ be the major index of w,
- $\mathrm{Des}(w) \ := \ \{j \in [n-1] : w(j) > w(j+1)\}$ be the descent set of w.

Motivation

Theorem (Foata–Schützenberger, 1978)

For every $J \subseteq [n-1]$ *and* $k \in \mathbb{N}$*, the number*

$$\#\{w \in \mathfrak{S}_n : \mathrm{Des}(w^{-1}) = J,\ \mathrm{inv}(w) = k\}$$

is equal to

$$\#\{w \in \mathfrak{S}_n : \mathrm{Des}(w^{-1}) = J,\ \mathrm{maj}(w) = k\}.$$

Motivation

Let $\mathcal{E}_n$ be the set of permutations (desarrangements) $w \in \mathfrak{S}_n$ for which the minimum element of $[n] \smallsetminus \mathrm{Des}(w)$ is even.

Theorem (Désarménien–Wachs, 1988)

For every $J \subseteq [n-1]$,

$$\#\{w \in \mathcal{D}_n : \mathrm{Des}(w) = J\} \;=\; \#\{w \in \mathcal{E}_n : \mathrm{Des}(w^{-1}) = J\}.$$

Note: A B_n-analogue of the theorem of Foata–Schützenberger was given by Adin–Brenti–Roichman (2006) and another by Foata–Han (2007).

Problem: Find a B_n-analogue of the theorem of Désarménien–Wachs.

Motivation

These two theorems are related to the representation theory of $\mathfrak{S}_n$. For instance, the original proof of Désarménien–Wachs showed that

$$\sum_{w \in \mathcal{D}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{w \in \mathcal{E}_n} F_{n,\mathrm{Des}(w^{-1})}(x)$$

and that the two handsides are in fact symmetric functions in x, where for $S \subseteq [n-1]$

$$F_{n,S}(x) \;=\; \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

is Gessel's fundamental quasisymmetric function.

Moreover, Reiner–Webb (2004) found a natural $\mathfrak{S}_n$-representation whose Frobenius characteristic is the Désarménien–Wachs symmetric function. More precisely, they showed that

$$\mathrm{ch}(\varepsilon_n \otimes \chi_n) \;=\; \sum_{w \in \mathcal{D}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{w \in \mathcal{E}_n} F_{n,\mathrm{Des}(w^{-1})}(x)$$

where ε_n is the sign character and

$$\chi_n \;=\; \sum_{k=0}^{n} (-1)^{n-k} \; 1\uparrow_{(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n}$$

is the character of the natural $\mathfrak{S}_n$-representation on the top homology of the complex of injective words.

Motivation

The Foata–Schützenberger theorem has a representation-theoretic proof via the theory of character formulas of Adin–Roichman (2015), which we now explain.

Roichman's formula

Recall that the irreducible characters of the symmetric group $\mathfrak{S}_n$ over $\mathbb{C}$ can be indexed by partitions of n. Let

- χ^λ be the irreducible $\mathfrak{S}_n$-character associated to $\lambda \vdash n$,
- $\mathrm{SYT}(\lambda)$ be the set of SYT of shape λ

and for $\lambda \vdash n$ and $Q \in \mathrm{SYT}(\lambda)$ let

- $\mathrm{Des}(Q)$ be the set of entries $j \in [n-1]$ of Q for which $j+1$ appears in a lower row than j.

Example: $\mathrm{Des}\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} \right) = \{2, 5\}.$

Roichman's formula

The concept of unimodality of a set with respect to a composition will be useful. Let

- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of n,
- $S(\alpha) = \{r_1, r_2, \ldots, r_k\}$, where $r_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$ and $r_0 = 0$,
- $J \subseteq [n-1]$.

Consider the segments

$$\{r_{i-1}+1, r_{i-1}+2, \ldots, r_i - 1\}, \qquad 1 \leq i \leq k.$$

We call J α-unimodal if its intersection with each segment is a prefix, possibly empty, of that segment for all $1 \leq i \leq k$.

Roichman's formula

Example

Let $\alpha = (3, 1, 4, 2)$. Then $S(\alpha) = \{3, 4, 8, 10\}$, the segments are

$$\{1, 2\},\ \varnothing,\ \{5, 6, 7\},\ \{9\}$$

and hence $\{1, 3, 5, 6\}$ is α-unimodal but $\{1, 3, 5, 7\}$ is not.

Roichman's formula

Theorem (Roichman 1997, Fomin–Greene 1998)

For all partitions $\lambda \vdash n$ and compositions $\alpha \models n$,

$$\chi^\lambda(\alpha) = \sum_{Q \in \mathrm{SYT}(\lambda)} \mathrm{wt}_\alpha(\mathrm{Des}(Q)),$$

where

$$\mathrm{wt}_\alpha(J) := \begin{cases} 0, & \text{if } J \text{ is not } \alpha\text{-unimodal;} \\ (-1)^{|J \setminus S(\alpha)|}, & \text{otherwise} \end{cases}$$

for $J \subseteq [n-1]$.

Fine sets for $\mathfrak{S}_n$

Definition (Adin–Roichman, 2015)

Let χ be an $\mathfrak{S}_n$-character. A fine set for χ is a set $\mathcal{B}$, endowed with a map $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$, such that

$$\chi(\alpha) = \sum_{b \in \mathcal{B}} \mathrm{wt}_\alpha(\mathrm{Des}(b))$$

for every composition α of n, where

$$\mathrm{wt}_\alpha(J) = \begin{cases} 0, & \text{if } J \text{ is not } \alpha\text{-unimodal;} \\ (-1)^{|J \setminus S(\alpha)|}, & \text{otherwise} \end{cases}$$

for $J \subseteq [n-1]$.

Fine sets for $\mathfrak{S}_n$

Example

The set $\mathrm{SYT}(\lambda)$, endowed with the standard descent map

$$\mathrm{Des} : \mathrm{SYT}(\lambda) \to 2^{[n-1]},$$

is a fine set for χ^λ for every $\lambda \vdash n$.

Fine sets for $\mathfrak{S}_n$

Other examples of $\mathfrak{S}_n$-characters and corresponding combinatorial objects giving rise to fine sets include:

- Gelfand models and involutions in $\mathfrak{S}_n$,
- coinvariant algebra characters and permutations of given inversion number,
- Lie characters and conjugacy classes (Gessel-Reutenauer, 1993),
- characters of Specht modules of zigzag shapes and inverse descent classes,
- certain induced characters and k-roots of the identity,
- characters induced from exterior algebras and arc permutations (Elizalde-Roichman, 2014).

Fine sets for $\mathfrak{S}_n$

Theorem (Adin–Roichman, 2015)

Given a fine set $\mathcal{B}$ for an $\mathfrak{S}_n$-character χ, the distribution of Des over $\mathcal{B}$ is uniquely determined by χ.

Corollary (Adin–Roichman, 2015)

The theorem of Foata–Schützenberger can be derived from (and is in fact equivalent to) one of Lusztig and Stanley on the representation of $\mathfrak{S}_n$ on its coinvariant algebra.

Fine sets for $\mathfrak{S}_n$

Recall that the Frobenius characteristic of a class function $\chi : \mathfrak{S}_n \to \mathbb{C}$ is defined by setting

$$\mathrm{ch}(\chi^\lambda) = s_\lambda(x)$$

and extending by linearity.

Theorem (Adin–A–Elizalde–Roichman)

A set $\mathcal{B}$, endowed with a map $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$, is fine for a character χ of $\mathfrak{S}_n$ if and only if

$$\mathrm{ch}(\chi) = \sum_{b \in \mathcal{B}} F_{n,\mathrm{Des}(b)}(x).$$

In particular, the distribution of Des over $\mathcal{B}$ is uniquely determined by χ.

Fine sets for $\mathfrak{S}_n$

For instance, from the formula

$$\mathrm{ch}(\varepsilon_n \otimes \chi_n) \;=\; \sum_{w \in \mathcal{D}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{w \in \mathcal{E}_n} F_{n,\mathrm{Des}(w^{-1})}(x)$$

of Désarménien–Wachs and Reiner–Webb we deduce:

Corollary

The sets $\mathcal{D}_n$ and $\{w^{-1} : w \in \mathcal{E}_n\}$ are both fine for the sign twist of the Reiner–Webb character χ_n. In particular,

$$\chi_n(\alpha) \;=\; \varepsilon_\alpha \sum_{w \in \mathcal{D}_n} \mathrm{wt}_\alpha(w)$$

for every composition α of n.

B_n-analogue of Roichman's formula

Our goal is to extend this theory to the group of signed permutations

$$B_n \ = \ \{w = (w(1), w(2), \ldots, w(n)) : \, |w| \in \mathfrak{S}_n\}.$$

Note: To find the right B_n-analogue of the concept of fine set, we need to find the right B_n-analogue of Roichman's formula for the irreducible characters of $\mathfrak{S}_n$.

B_n-analogue of Roichman's formula

Recall that the irreducible characters of the hyperoctahedral group B_n over $\mathbb{C}$ can be indexed by bipartitions of n, meaning pairs (λ, μ) of partitions of total sum n. We need to replace:

- partitions $\lambda \vdash n$ by bipartitions $(\lambda, \mu) \vdash n$,
- SYT of shape λ by SY bitableaux (Q^+, Q^-) of shape (λ, μ),

$$\left(\begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 4 & 9 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \, , \quad \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 7 & 8 \\ \hline \end{array} \right)$$

- compositions of n and subsets of $[n-1]$ by signed compositions of n and signed subsets of $[n]$,
- descent sets of SYT by signed descent sets of SY bitableaux,
- the weight of a set with respect to a composition by that of a signed set with respect to a signed composition.

B_n-analogue of Roichman's formula

A signed subset of $[n]$ is a pair (J, ε) where

- $J \subseteq [n]$ contains n and
- $\varepsilon : J \to \{-, +\}$ is a map.

Note: Suppose $J = \{s_1 < s_2 < \cdots < s_k\}$, where $s_k = n$, and set $s_0 := 0$. The map ε can be extended to a sign vector $\varepsilon : [n] \to \{-, +\}^n$ by setting

$$\varepsilon(j) = \varepsilon(s_i)$$

for $s_{i-1} < j \leq s_i$ and $1 \leq i \leq k$.

Note: The signed subsets of $[n]$ are in one-to-one correspondence with the signed compositions of n, meaning compositions of n for which each part has been assigned the positive or the negative sign.

B_n-analogue of Roichman's formula

Example

Let $n = 9$ and $J = \{3, 5, 6, 8, 9\}$ with

$$\varepsilon(3) = -,\ \varepsilon(5) = \varepsilon(6) = +,\ \varepsilon(8) = -,\ \varepsilon(9) = +.$$

Then (J, ε) is a signed set with corresponding signed composition

$$(3^-, 2^+, 1^+, 2^-, 1^+)$$

and sign vector

$$(-, -, -, +, +, +, -, -, +).$$

B_n-analogue of Roichman's formula

The signed (or colored) descent set of $w = (w(1), w(2), \ldots, w(n)) \in B_n$ is the pair (J, ε) defined by letting

$$\varepsilon_i \;=\; \begin{cases} +, & \text{if } w(i) > 0; \\ -, & \text{if } w(i) < 0 \end{cases}$$

and $J \subseteq [n]$ consist of n along with all $j \in [n-1]$ for which

- $w(j)$ and $w(j+1)$ have different signs, or
- $w(j) > w(j+1) > 0$, or
- $-w(j) > -w(j+1) > 0$.

B_n-analogue of Roichman's formula

Example

Let $n = 9$, $w = (-2, -5, -7, 3, 8, 1, -4, -9, 6)$. Then

$$\varepsilon \;=\; (-,\, -,\, -,\, +,\, +,\, +,\, -,\, -,\, +), \qquad J \;=\; \{3, 5, 6, 8, 9\}$$

with corresponding signed composition

$$(3^-,\, 2^+,\, 1^+,\, 2^-,\, 1^+).$$

B_n-analogue of Roichman's formula

The signed (or colored) descent set (J, ε) of a SY bitableau (Q^+, Q^-) of shape $(\lambda, \mu) \vdash n$ is defined by letting

$$\varepsilon_i = \begin{cases} +, & \text{if } i \text{ appears in } Q^+; \\ -, & \text{if } i \text{ appears in } Q^- \end{cases}$$

and letting $J \subseteq [n]$ consist of n along with all $j \in [n-1]$ for which

- j appears in Q^+ and $j+1$ in Q^-, or
- j appears in Q^- and $j+1$ in Q^+, or
- j and $j+1$ appear in the same tableau and $j+1$ appears in a lower row than j.

B_n-analogue of Roichman's formula

Example

Let $n = 9$ and $Q = (Q^+, Q^-)$, where

$$Q^+ = \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 4 & 9 \\ \cline{1-2}\end{array}\,, \qquad Q^- = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 7 & 8 \\ \hline\end{array}\,.$$

Then

$$\varepsilon = (-, +, +, +, +, -, -, -, +), \qquad J = \{1, 3, 5, 6, 8, 9\}$$

with corresponding signed composition

$$(1^-, 2^+, 2^+, 1^-, 2^-, 1^+).$$

B_n-analogue of Roichman's formula

We let

- $\Sigma^B(n)$ be the set of signed subsets of $[n]$,
- $\chi^{\lambda,\mu}$ be the irreducible B_n-character associated to $(\lambda, \mu) \vdash n$,
- $\mathrm{SYT}(\lambda, \mu)$ be the set of SY bitableaux of shape $(\lambda, \mu) \vdash n$,
- $\mathrm{cDes}(w)$ be the signed descent set of $w \in B_n$,
- $\mathrm{cDes}(Q)$ be the signed descent set of $Q \in \mathrm{SYT}(\lambda, \mu)$,

so that $\mathrm{cDes}(w), \mathrm{cDes}(Q) \in \Sigma^B(n)$.

B_n-analogue of Roichman's formula

Theorem (Adin–A–Elizalde–Roichman)

For all bipartitions $(\lambda, \mu) \vdash n$ *and signed compositions* γ *of* n*,*

$$\chi^{\lambda,\mu}(\gamma) \;=\; \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} \mathrm{wt}_\gamma(\mathrm{cDes}(Q)),$$

where $\mathrm{wt}_\gamma(\sigma)$ *is defined in the sequel.*

B_n-analogue of Roichman's formula

To define the weight function wt_γ let

- γ be a signed composition of n,
- $S(\gamma) = S(|\gamma|) = \{r_1, r_2, \ldots, r_k\}$ and set $r_0 = 0$,
- $\sigma = (J, \varepsilon) \in \Sigma^B(n)$

and consider again the segments $\{r_{i-1}+1, r_{i-1}+2, \ldots, r_i - 1\}$ for $1 \leq i \leq k$. Then

$$\mathrm{wt}_\gamma(\sigma) := \begin{cases} 0, & \text{if } J \text{ is not } |\gamma|\text{-unimodal;} \\ 0, & \text{if } \sigma \text{ assigns different signs to two} \\ & \text{elements of the same segment;} \\ (-1)^{|J \smallsetminus S(\gamma)| + n_\gamma(\sigma)}, & \text{otherwise} \end{cases}$$

where $n_\gamma(\sigma)$ is the number of segments which are assigned the negative sign by both σ and γ.

Fine sets for B_n

Definition

Let χ be a B_n-character. A fine set for χ is a set $\mathcal{B}$, endowed with a map $\mathrm{cDes} : \mathcal{B} \to \Sigma^B(n)$, such that

$$\chi(\gamma) \; = \; \sum_{b \in \mathcal{B}} \mathrm{wt}_\gamma(\mathrm{cDes}(b))$$

for every signed composition γ of n.

Fine sets for B_n

Example

The set $\mathrm{SYT}(\lambda, \mu)$, endowed with the standard signed descent map

$$\mathrm{cDes} : \mathrm{SYT}(\lambda, \mu) \to \Sigma^B(n),$$

is a fine set for $\chi^{\lambda,\mu}$ for every bipartition $(\lambda, \mu) \vdash n$.

Fine sets for B_n

The following signed analogues of the functions $F_{n,S}(x)$ were introduced and studied by Poirier (1998). For $\sigma = (J, \varepsilon) \in \Sigma^B(n)$ define

$$F_\sigma(x, y) = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in \mathrm{Des}(\sigma) \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}$$

where

$$z_i = \begin{cases} x_i, & \text{if } \varepsilon_i = +; \\ y_i, & \text{if } \varepsilon_i = - \end{cases}$$

and $\mathrm{Des}(\sigma)$ stands for the set of elements of $J \cap [n-1]$ except for those of negative sign immediately followed by one of positive sign.

Fine sets for B_n

Example

For $n = 6$ and $J = \{2, 3, 5, 6\}$ with sign vector $\varepsilon = (+, +, -, -, -, +)$,

$$F_\sigma(x, y) \;=\; \sum_{1 \le i_1 \le i_2 < i_3 < i_4 \le i_5 \le i_6} x_{i_1} x_{i_2} y_{i_3} y_{i_4} y_{i_5} x_{i_6}.$$

Fine sets for B_n

Recall that the Frobenius characteristic of a class function $\chi : B_n \to \mathbb{C}$ is defined by setting

$$\mathrm{ch}(\chi^{\lambda,\mu}) = s_\lambda(x) s_\mu(y)$$

and extending by linearity.

Theorem (Adin–A–Elizalde–Roichman)

A set $\mathcal{B}$, endowed with a map $\mathrm{cDes} : \mathcal{B} \to \Sigma^B(n)$, is fine for a character χ of B_n if and only if

$$\mathrm{ch}(\chi) = \sum_{b \in \mathcal{B}} F_{\mathrm{cDes}(b)}(x, y).$$

In particular, the distribution of cDes over $\mathcal{B}$ is uniquely determined by χ.

Examples

Other examples of B_n-characters and corresponding combinatorial objects giving rise to fine sets include:

- Gelfand models and involutions in B_n,
- coinvariant algebra characters and signed permutations of given flag inversion number,
- Lie characters and conjugacy classes (Poirier, 1998),
- signed Reiner–Webb characters and derangements in B_n,
- certain induced characters and k-roots of the identity,
- characters induced from exterior algebras and signed arc permutations.

Coinvariant algebra and flag statistics

Let F be a field of characteristic zero. The group B_n acts on the polynomial ring $F[x_1, x_2, \ldots, x_n]$ by permuting the variables and switching their signs. Let

- $P_n = F[x_1, x_2, \ldots, x_n]$,
- I_n^B be the ideal of P_n generated by the B_n-invariant polynomials of zero constant term,
- P_n/I_n^B be the coinvariant algebra of B_n.

The algebra P_n/I_n^B is graded by degree and B_n acts on each homogeneous component. Let

- $\chi_{n,k}^B$ be the character of the B_n-action on the kth homogeneous component of P_n/I_n^B.

Coinvariant algebra and flag statistics

The flag-major index of $w \in B_n$ is defined as

$$\operatorname{fmaj}(w) \;=\; 2 \sum_{i \in \operatorname{Des}(w)} i \;+\; \operatorname{neg}(w),$$

where $\operatorname{neg}(w)$ is the number of $i \in [n]$ with $w(i) < 0$. The flag-inversion number of $w \in B_n$ is defined as

$$\operatorname{finv}(w) \;=\; 2 \cdot \operatorname{inv}(w) \;+\; \operatorname{neg}(w),$$

where $\operatorname{inv}(w)$ is the number of inversions of $w = (w(1), w(2), \ldots, w(n))$ with respect to the total order

$$-1 \,<\, -2 \,<\, \cdots \,<\, -n \,<\, 1 \,<\, 2 \,<\, \cdots \,<\, n.$$

Coinvariant algebra and flag statistics

Theorem (Adin–A–Elizalde–Roichman)

The following subsets of B_n are both fine sets for the B_n-character $\chi^B_{n,k}$:

- $\{w \in B_n : \mathrm{finv}(w) = k\}$,
- $\{w \in B_n : \mathrm{fmaj}(w^{-1}) = k\}$.

Corollary (Foata–Han, 2007)

For every $\sigma \in \Sigma^B(n)$ and every $k \in \mathbb{N}$, the number

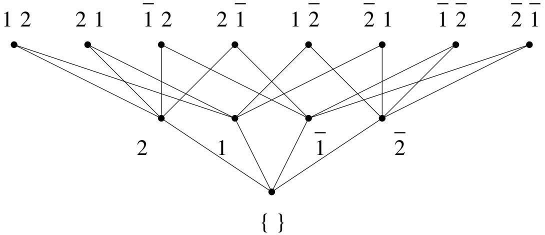
$$\#\{w \in B_n : \ \mathrm{cDes}(w) = \sigma, \ \mathrm{finv}(w) = k\}$$

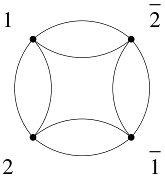
is equal to

$$\#\{w \in B_n : \ \mathrm{cDes}(w) = \sigma, \ \mathrm{fmaj}(w^{-1}) = k\}.$$

Derangements and desarrangements

Let ψ_n the character of the natural B_n-action on the top homology of the complex of injective words of type B_n.





Equivalently,

$$\psi_n \ = \ \sum_{k=0}^{n} (-1)^{n-k} \ 1\uparrow_{(\mathfrak{S}_1)^k \times B_{n-k}}^{B_n} .$$

Derangements and desarrangements

Let

- $\mathcal{D}_n^B$ be the set of derangements in B_n,
- $\mathcal{E}_n^B$ be the set of $w \in B_n$ for which the maximum k with

$$w(1) > w(2) > \cdots > w(k) > 0$$

  is even (possibly zero).

For instance,

- $\mathcal{D}_2^B = \{(-1,-2), (2,1), (-2,1), (2,-1), (-2,-1)\}$,
- $\mathcal{E}_2^B = \{(2,1), (-1,2), (-1,-2), (-2,1), (-2,-1)\}$.

Derangements and desarrangements

Theorem (Adin–A–Elizalde–Roichman)

For every positive integer n,

$$\omega_x \, \mathrm{ch}(\psi_n) \;=\; \sum_{w \in \mathcal{D}_n^B} F_{\mathrm{cDes}(w)}(x,y) \;=\; \sum_{w \in \mathcal{E}_n^B} F_{\mathrm{cDes}(w^{-1})}(x,y).$$

In particular,

$$\#\{w \in \mathcal{D}_n^B \,:\, \mathrm{cDes}(w) = \sigma\} \;=\; \#\{w \in \mathcal{E}_n^B \,:\, \mathrm{cDes}(w^{-1}) = \sigma\}$$

for every $\sigma \in \Sigma^B(n)$.

Open problems

- Characterize fine subsets of $\mathfrak{S}_n$ and B_n or, equivalently, subsets $\mathcal{B}$ for which

$$\sum_{b \in \mathcal{B}} F_{n,\mathrm{Des}(b)}(x) \quad \text{or} \quad \sum_{b \in \mathcal{B}} F_{\mathrm{cDes}(b)}(x, y),$$

  respectively, is a symmetric and Schur-nonnegative symmetric function.
- Find conceptual proofs of the two main results.
- Extend to other Coxeter groups and complex reflection groups.
- Extend to the Hecke algebras of $\mathfrak{S}_n$ and B_n.
- Give a bijective proof of the last statement on the previous page.