Maximal increasing sequences in fillings of almost-moon polyominoes

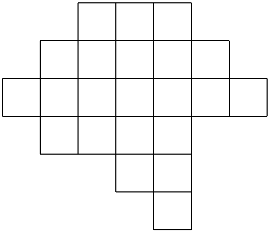
Svetlana Poznanović
joint work with
Catherine Yan

Department of Mathematical Sciences
Clemson University

Svetlana Poznanović

Moon Polyominoes

- *convex* rows and columns
- *comparable* rows and columns



- Lengths of rows from top to bottom form a unimodal sequence.
- Every cell is filled with either 0 or 1.

- A k-ne-chain: A set of 1-cells $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}$ with $i_1 < \cdots < i_k, j_1 < \cdots < j_k$ such that the smallest rectangle containing them is a subset of $\mathcal{M}$.
- For a 01-filling M, $\mathrm{ne}(M)$ is the length of the largest ne-chain.

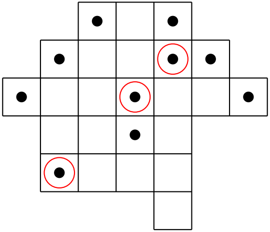
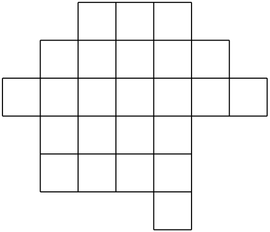
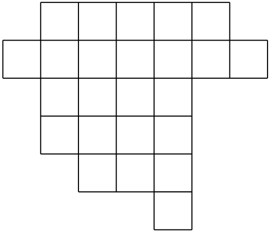


Figure: A 01-filling M of a moon polyomino with $\mathrm{ne}(M) = 3$. The 1's are represented by dots and the 0-cells are left empty. The circled dots form the only 3-chain in M.

For a moon polyomino $\mathcal{M}$, let $\sigma\mathcal{M}$ be another moon polyomino obtained by permuting the rows of M.





Theorem (Rubey, 2011)

The number of 01-fillings with the longest northeast chains of size k and exactly c_i non-zero entries in column i are equal for $\mathcal{M}$ and $\sigma\mathcal{M}$.

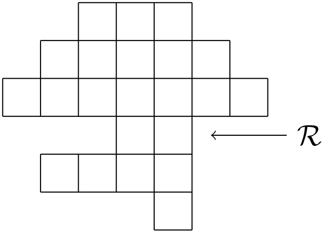
Brief history

- Chen, Deng, Du, Stanley, & Yan (2007) - set partitions
- Backelin, West & Xin (2007) + Krattenthaler (2006) + de Mier (2006) - Ferrers and reverse Ferrers shapes
- Jonsson (2007) + Jonsson & Welker (2007) - stack polyominoes
- Rubey (2011) - moon polyominoes for both 01 and $\mathbb{N}$ fillings
- Serrano & Stump (2012) - maximal 01-fillings of Ferrers shapes, bijective
- Rubey (2012) - maximal 01-fillings of stack polyominoes,bijective
- Bijective proof of Rubey's result for 01-fillings of moon polyominoes?

"If there is a problem you can't solve, then there is an easier problem you can solve: find it." - George Pólya

Almost-Moon Polyominoes

- *comparable* rows and columns
- *convex* rows
- at most one exceptional row



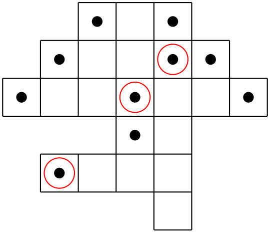
- A row $\mathcal{R}$ is an *exceptional row* of a polyomino $\mathcal{M}$ if there are *longer* rows both above $\mathcal{R}$ and below $\mathcal{R}$.

ne-chains in fillings of almost-moon polyominoes

- A k-ne-chain is a set of k cells $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\}$ with $i_1 < \cdots < i_k, j_1 < \cdots < j_k$ filled with 1's such that the $k \times k$ submatrix

$$\{(i_r,j_s) : 1 \leq r \leq k, 1 \leq s \leq k\}$$

is contained in the polyomino
(with no restriction on the filling of the other cells).



Explicit bijection ϕ

- $\mathcal{F}(\mathcal{M})$ = the set of all 01-fillings of a polyomino $\mathcal{M}$

Theorem (P & Yan 2015)

Let $\mathcal{M}$ and $\mathcal{N}$ be two almost-moon polyominoes such that $\mathcal{N}$ can be obtained from $\mathcal{M}$ by an interchange of two adjacent rows. In addition assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. Then there is a bijection

$$\phi_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{N})$$

that preserves the column sums of the fillings and such that

$$\mathrm{ne}(\phi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M) \quad \textit{for} \quad M \in \mathcal{F}(\mathcal{M}).$$

Moreover, $\phi_{\mathcal{N},\mathcal{M}} \circ \phi_{\mathcal{M},\mathcal{N}} = 1_{\mathcal{F}(\mathcal{M})}$.

Note

- $\mathcal{N} = \sigma\mathcal{M}$ then $\mathcal{N}$ can be obtained from $\mathcal{M}$ by several interchanges of adjacent rows so that the intermediate polyominoes are also almost-moon.
- Composing the maps from our theorem, we get a bijection that proves Rubey's result for almost-moon polyominoes.
- Our map preserves the column sums, but not the row sums (not even if they are 1). More about this later ...

Construction of $\phi(M)$: Step 1

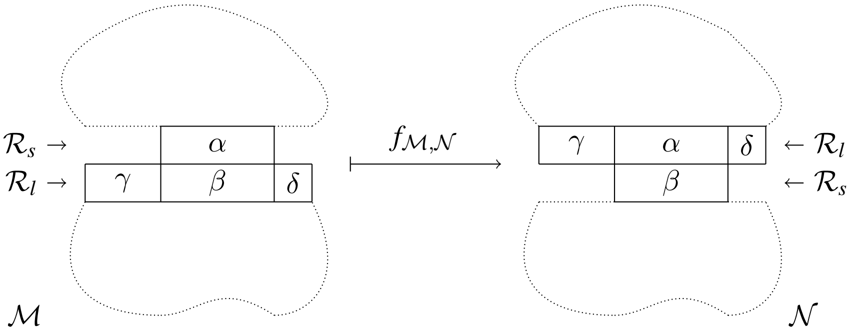


Figure: The fillings M and $f_{\mathcal{M},\mathcal{N}}(M)$ differ only in the rows $\mathcal{R}_s$ and $\mathcal{R}_l$.

Construction of $\phi(M)$: Step 1

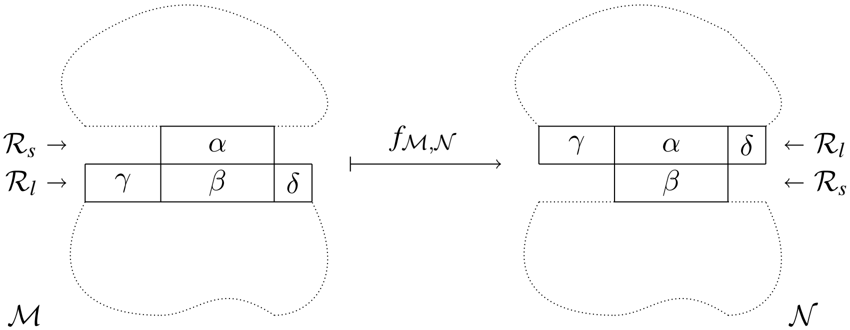
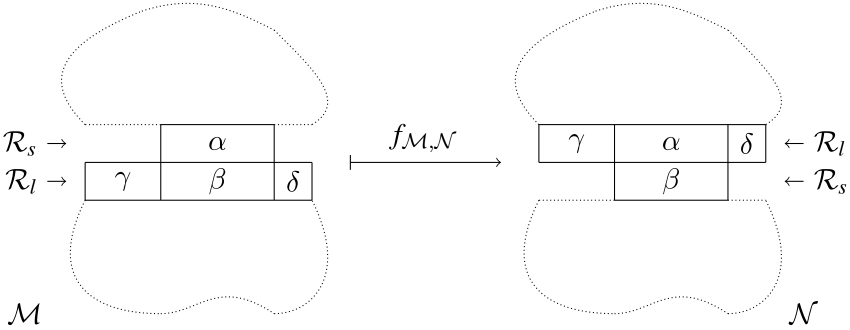


Figure: The fillings M and $f_{\mathcal{M},\mathcal{N}}(M)$ differ only in the rows $\mathcal{R}_s$ and $\mathcal{R}_l$.

Possible cases:

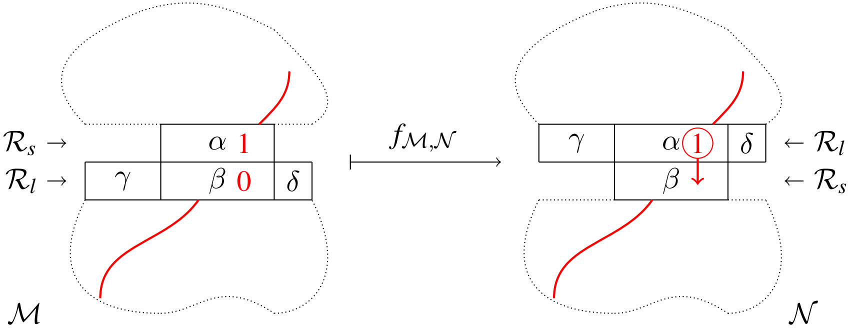
- $\text{ne}(f(M)) = \text{ne}(M)$
- $\text{ne}(f(M)) = \text{ne}(M) + 1$
- $\text{ne}(f(M)) = \text{ne}(M) - 1$

Construction of $\phi(M)$: Step 2



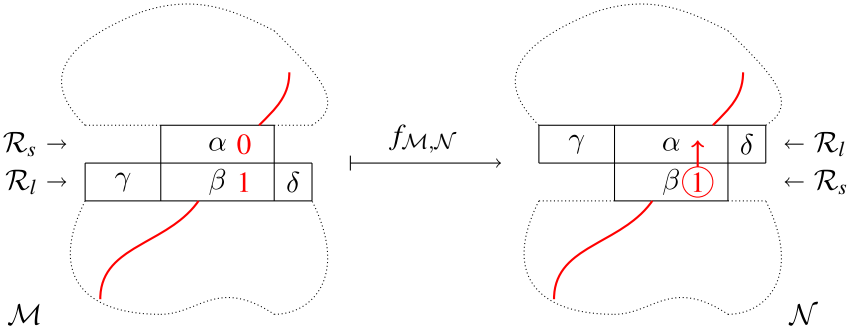
- If $\mathrm{ne}(f(M)) = \mathrm{ne}(M)$ then $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(M)$.

Construction of $\phi(M)$: Step 2



- If $\mathrm{ne}(M) = k$ and $\mathrm{ne}(f(M)) = k + 1$:
  - The 1-cells in the part α of $f(M)$ which are part of a $(k+1)$-chain are called *problem cells*.
  - $\phi_{\mathcal{M},\mathcal{N}}(M)$ is obtained by a vertical shift of the problem cells in $f_{\mathcal{M},\mathcal{N}}(M)$ from row $\mathcal{R}_l$ to row $\mathcal{R}_s$

Construction of $\phi(M)$: Step 2



- If $\mathrm{ne}(M) = k$ and $\mathrm{ne}(f(M)) = k - 1$:
  - The 1-cells in the part β of M which are part of a k-chain are called *problem cells*.
  - $\phi_{\mathcal{M},\mathcal{N}}(M)$ is obtained by a vertical shift of the problem cells in $f_{\mathcal{M},\mathcal{N}}(M)$ from row $\mathcal{R}_s$ to row $\mathcal{R}_l$

Remarks

The construction works! – the proof is quite technical.

- If $\mathcal{M}$ and $\mathcal{N}$ are related by permutation of rows but one of them is not an almost-moon polyomino then, in general,

$$\sum_{M \in \mathcal{F}(\mathcal{M})} q^{\mathrm{ne}(M)} \neq \sum_{M \in \mathcal{F}(\mathcal{N})} q^{\mathrm{ne}(M)}.$$

- Our map ϕ preserves the column sums. In general, ne is not equally distributed over $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(\sigma\mathcal{M})$ if one wants to preserve both column and row sums. But ...

Explicit bijection ψ for 0-1 row sums

- $\mathcal{F}(M, \mathbf{r}, \mathbf{c})$ = the set of all 01-fillings of a polyomino $\mathcal{M}$ with row sums given by the vector $\mathbf{r}$ and column sums given by the vector $\mathbf{c}$

Theorem (P & Yan 2015)

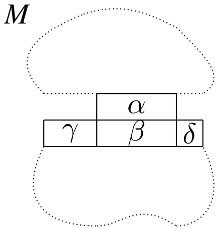
Let $\mathcal{M}$ and $\mathcal{N} = \sigma\mathcal{M}$ be two almost-moon polyominoes such that $\mathcal{N}$ can be obtained from $\mathcal{M}$ by an interchange of two adjacent rows. In addition assume that $\mathcal{M}$ and $\mathcal{N}$ have no exceptional rows other than the swapped ones. If $\boldsymbol{r} \in \{0,1\}^*$, then there is a bijection

$$\psi_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}) \longrightarrow \mathcal{F}(\mathcal{N}, \sigma\boldsymbol{r}, \boldsymbol{c})$$

such that

$$\mathrm{ne}(\psi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M) \quad \textit{for} \quad M \in \mathcal{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}).$$

Coupled fillings



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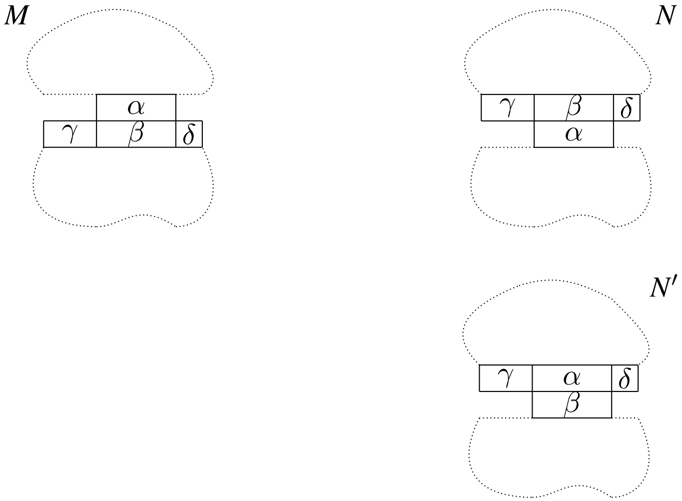
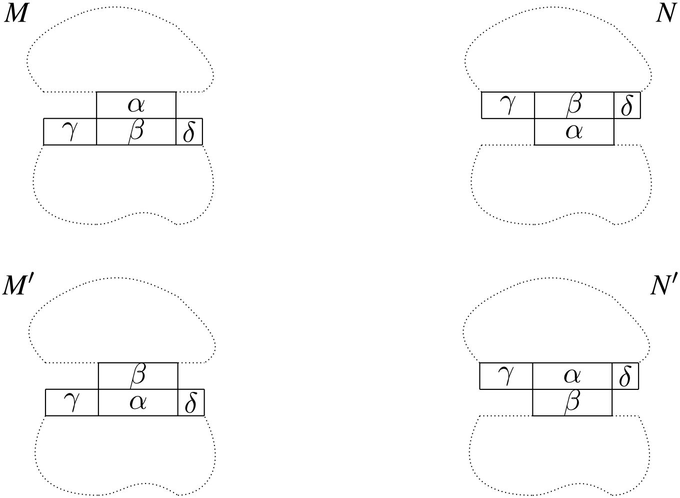


Coupled fillings



Definition of $\psi_{\mathcal{M},\mathcal{N}}$

Lemma

Let (M, M') be a pair of coupled fillings in $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ and (N, N') be the corresponding pair of coupled fillings of $\mathcal{N}$. Then

$$\{\mathrm{ne}(M), \mathrm{ne}(M')\} = \{\mathrm{ne}(N), \mathrm{ne}(N')\}$$

as multisets.

Definition (of the bijection)

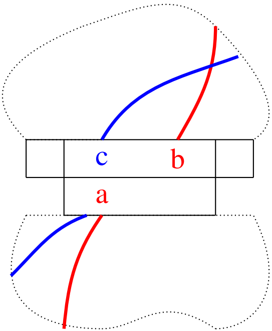
$$\psi_{\mathcal{M},\mathcal{N}}(M) = \begin{cases} N & \text{if } \mathrm{ne}(M) = \mathrm{ne}(N) \\ N' & \text{if } \mathrm{ne}(M) \neq \mathrm{ne}(N). \end{cases}$$

Proof of the lemma.

- Only need to show that $\mathrm{ne}(M) \in \{\mathrm{ne}(N), \mathrm{ne}(N')\}$.
- Technical, but ...

Idea of both proofs

If there is a $(k+1)$-chain and there is a $(k+1)$-chain in $\mathcal{N}$,



Then M has a $(k+1)$-chain not containing cell a.

Open questions

- Rubey showed that maximal 01-fillings of moon polyominoes with restricted chain lengths can be identified with certain rc-graphs, also known as pipe dreams. Is there an extension of pipe dreams to almost-moon polyominoes?

- Is there a relationship between the Edelman-Greene correspondence and the map on moon polyominoes induced by the $\phi_{\mathcal{M},\mathcal{N}}$'s? The analogous question for the Robinson-Schensted map and the BWX map (for Ferrers shapes) was answered by Bloom & Saracino.

- Is there a "nice" bijection that reverses the sequence of the rows of the polyomino?

Thank you very much!