The frequency of pattern occurrence in random walks

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Dartmouth College

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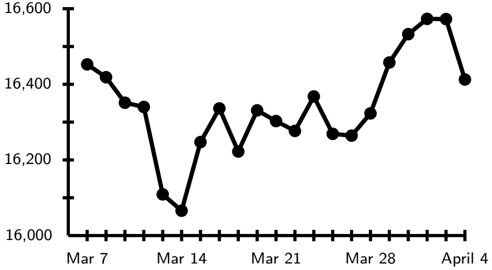
Ordinal Patterns in Time Series

Dow Jones Industrial Average: March 7, 2014 - April 4, 2014



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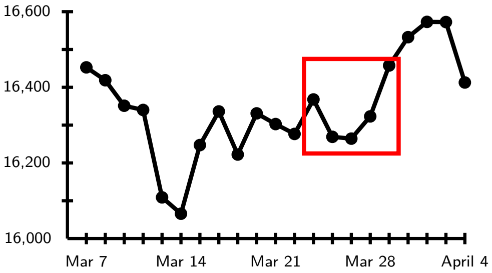

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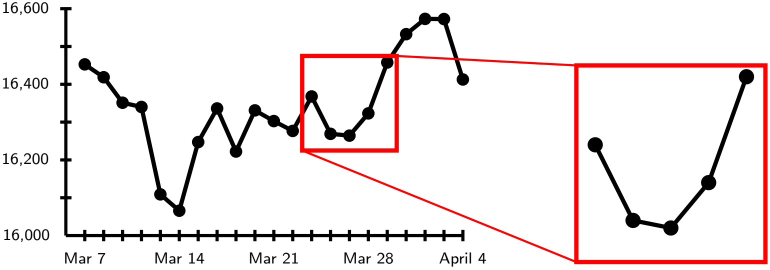

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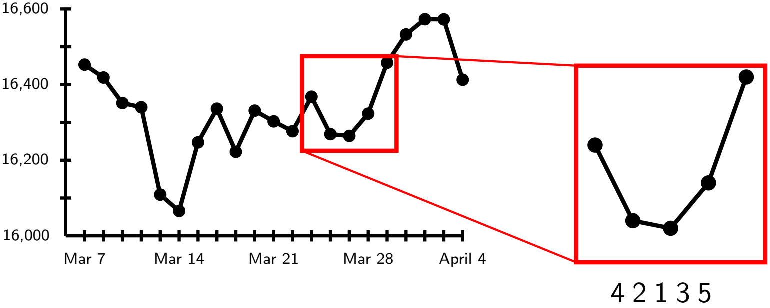

Ordinal Patterns in Time Series Analysis

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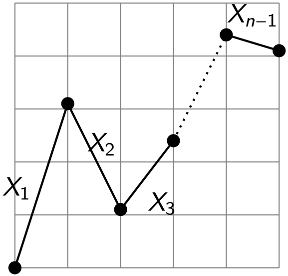
- Frequency of pattern occurrence in random settings?

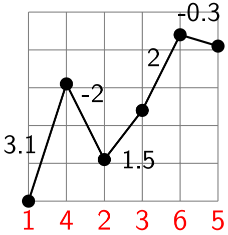
Random Walks

- Random Walk on the real number line.
- $X_1, X_2, \ldots, X_{n-1}$ independent and identically distributed random variables (i.i.d.)
- At time i, the walker is at $X_1 + X_2 + \ldots + X_i$.



Patterns in Random Walks

- A random walk with $n-1$ steps corresponds to a pattern of length n.



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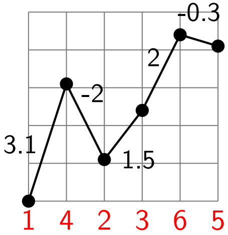

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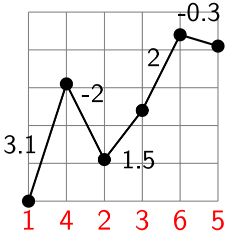

- $p(3.1, -2, 1.5, 2, -0.3) = 142365$
- In general, $p(x_1, x_2, \ldots, x_{n-1}) = \pi$ if $x_1, x_2, \ldots, x_{n-1}$ are the steps for a walk with pattern π.

Probabilities of Pattern Occurrence

- Pattern occurrence in a length 1,000,000 random walk.
- Gaussian distribution with mean=1 and standard deviation=4.

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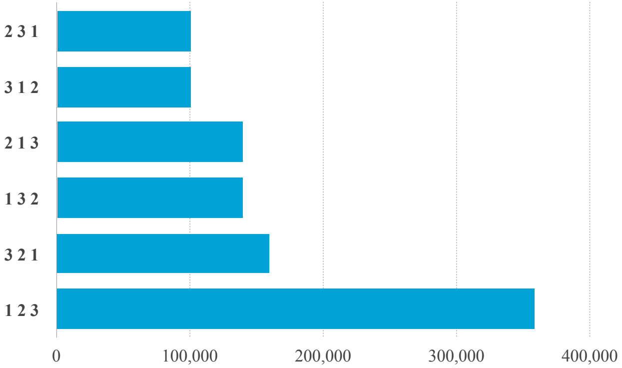

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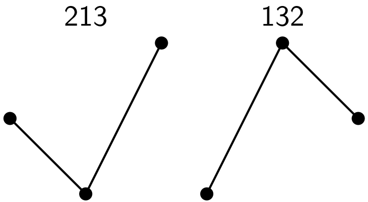

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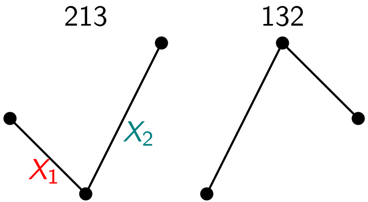

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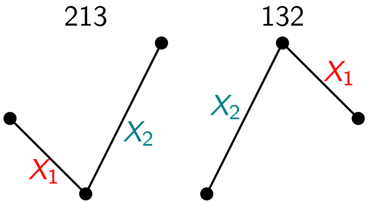

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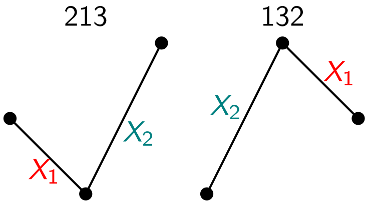

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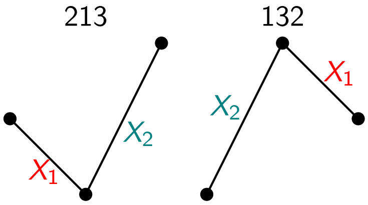

- 213 and 132 have the same probability of occurring regardless of distribution!
- For what other patterns does this phenomenon occur?

An Equivalence on Patterns

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$$\boxed{\pi \sim \tau}$$

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For some $\rho \in S_{n-1}$

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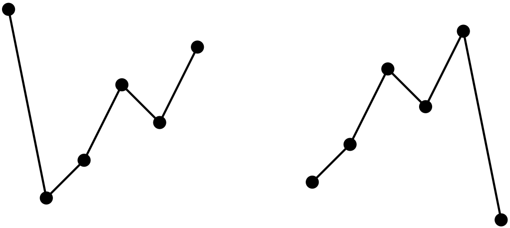

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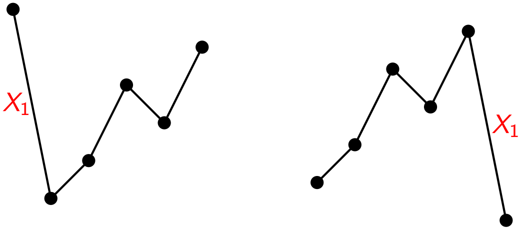

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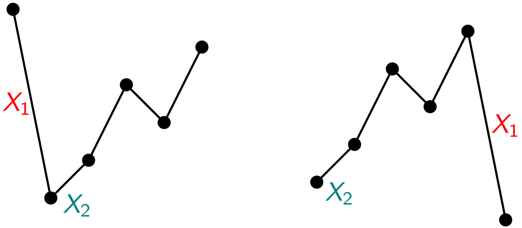

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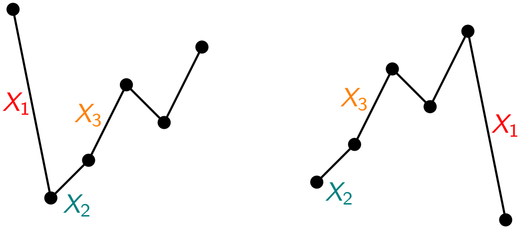

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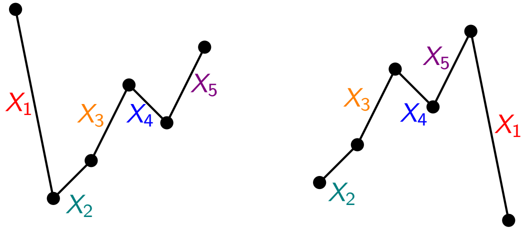

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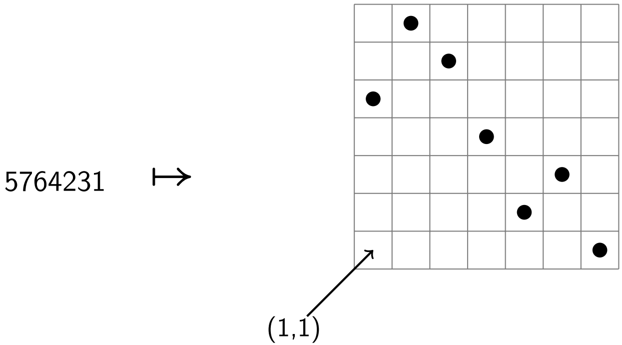
Conjecture

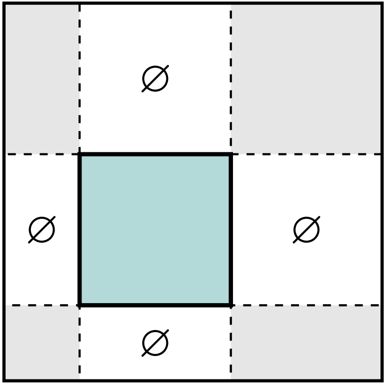
$\mathbb{P}(p(X_1, X_2, \ldots, X_{n-1}) = \pi) = \mathbb{P}(p(X_1, X_2, \ldots, X_{n-1}) = \tau)$ *for every probability distribution on the i.i.d. random variables* $X_1, X_2, \ldots, X_{n-1}$ *if and only if* $\pi \sim \tau$.

Patterns on Grids



- Fill cell $(i, \pi(i))$ for all $i \in [n]$.

Bordered Cylindrical Blocks



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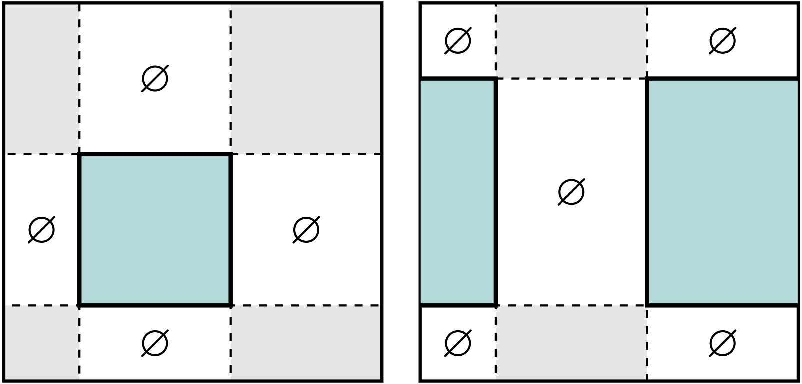

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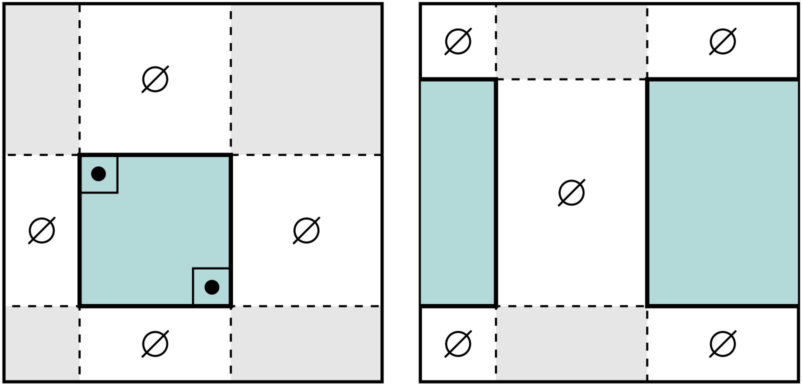

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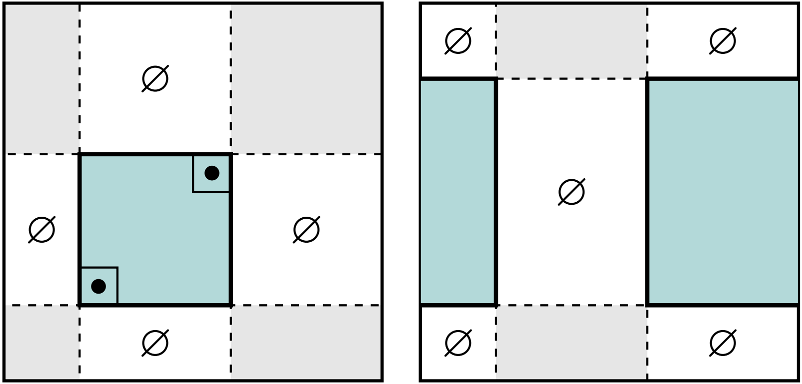

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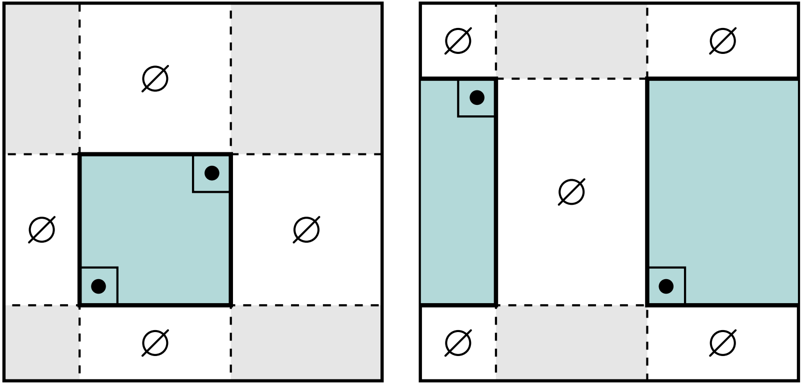

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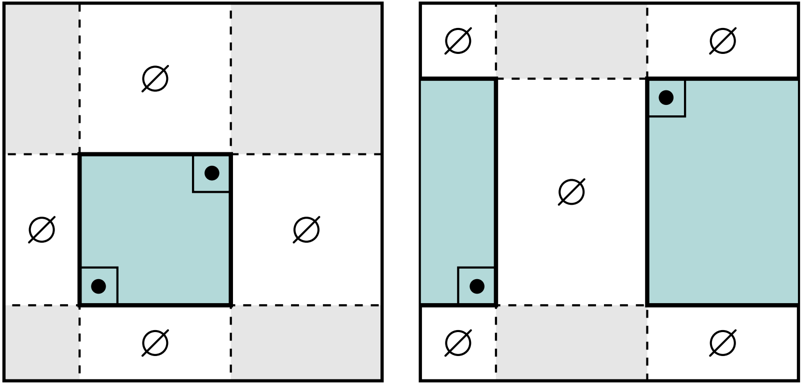
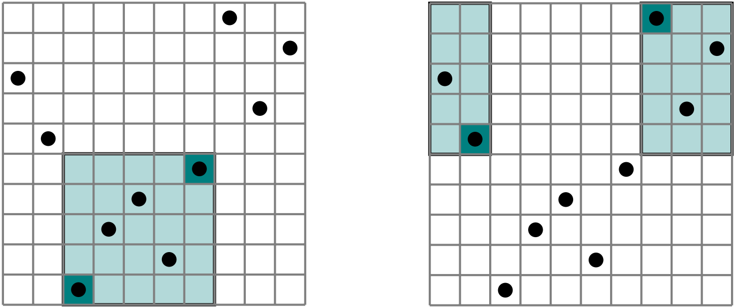
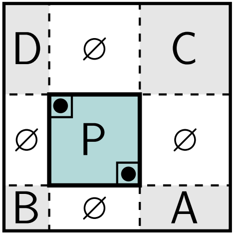


Examples of Bordered Cylindrical Blocks



Flipping Bordered Cylindrical Blocks



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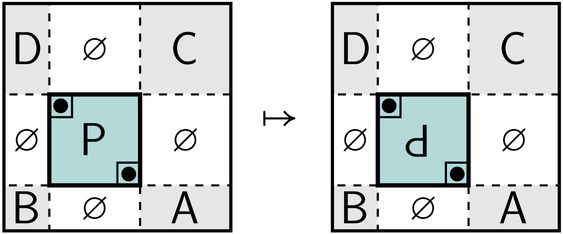

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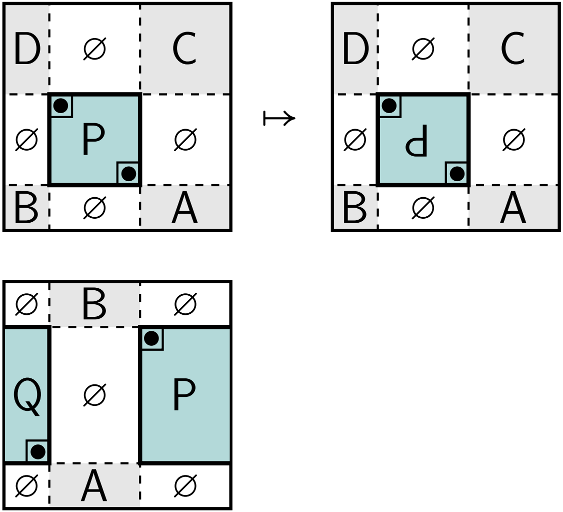

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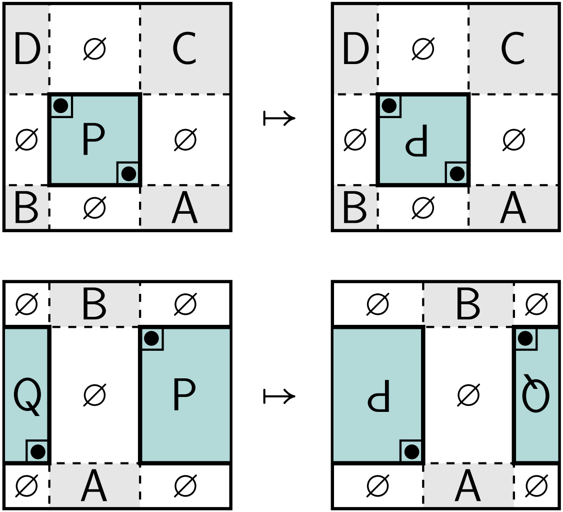

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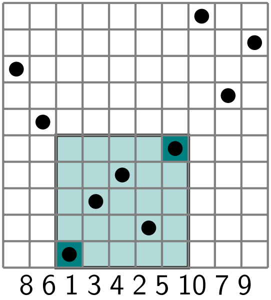

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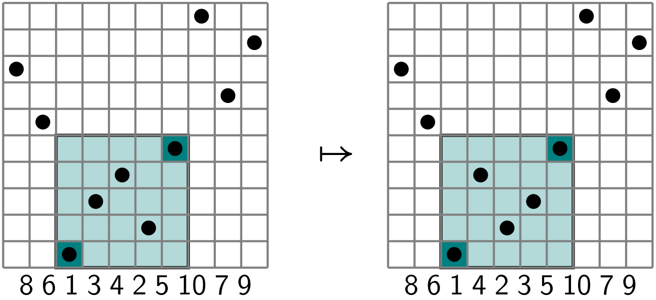

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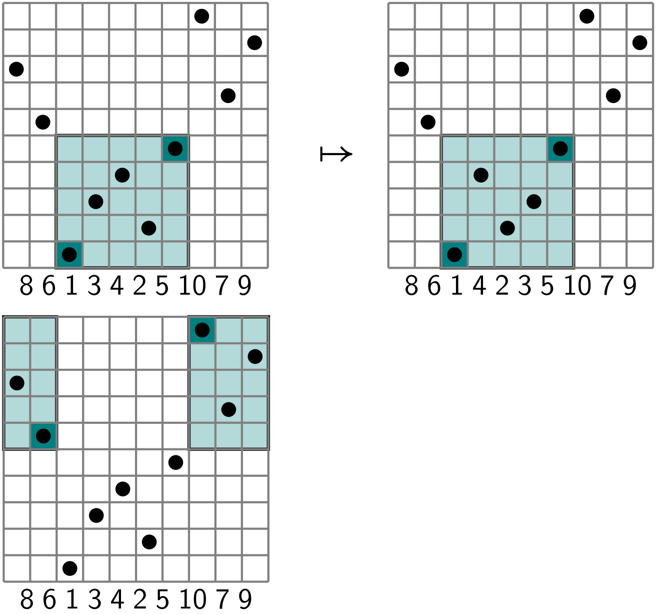

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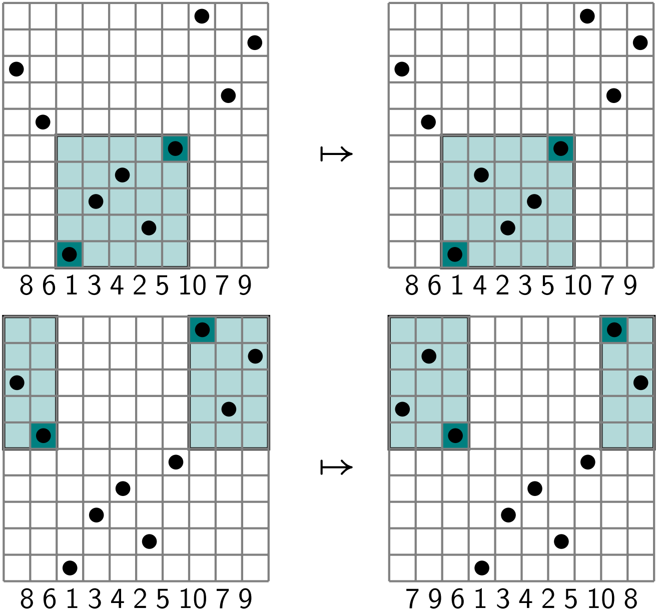

Characterizing Equivalence Classes

Theorem

$\pi \sim \tau$ if and only if τ can be obtained from π by a sequence of flips of bordered cylindrical blocks and the reverse-complement operation.

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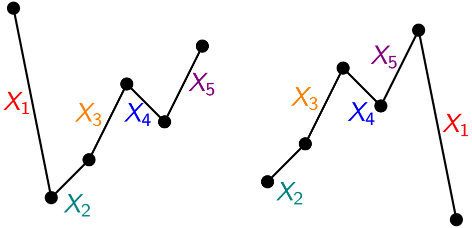
$612435 \sim 235461$



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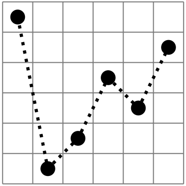

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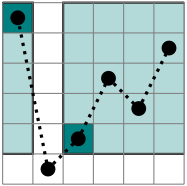

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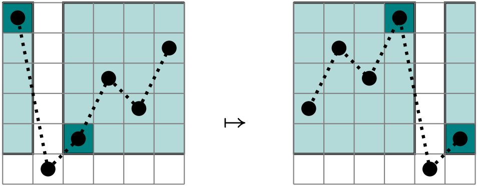

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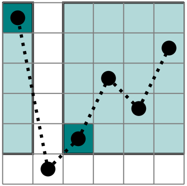

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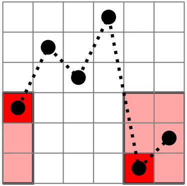

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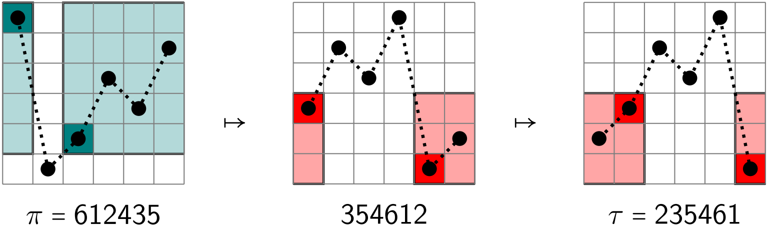
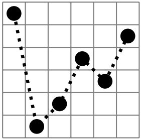


The Entire Equivalence Class



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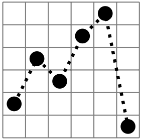

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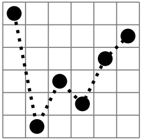

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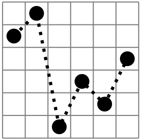

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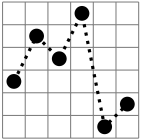

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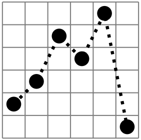

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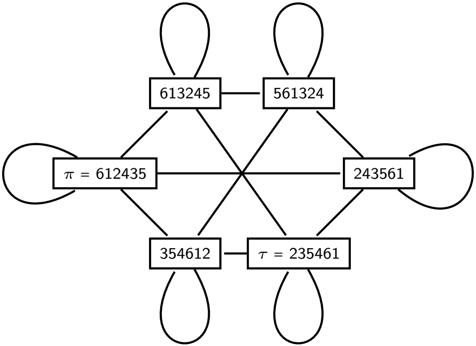
Equivalence Graphs



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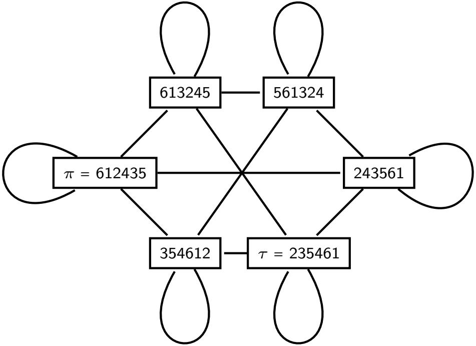

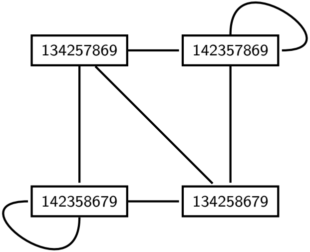

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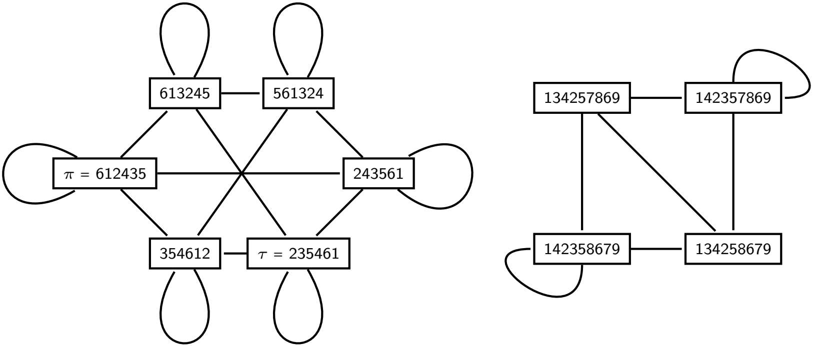
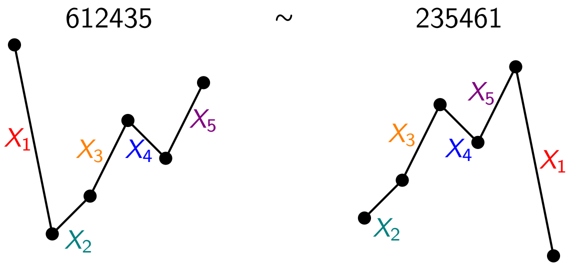


Proposition

If $\pi \sim \tau$ than π and τ contain the same number of bordered cylindrical blocks.

Motivation



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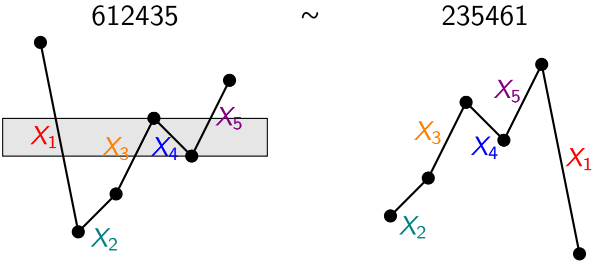

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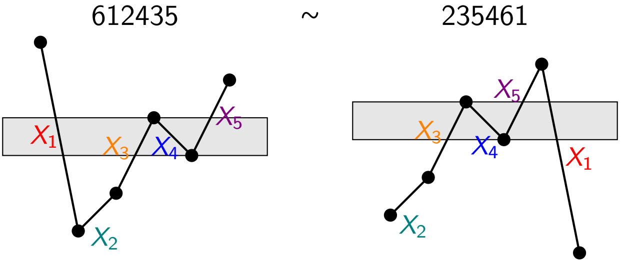
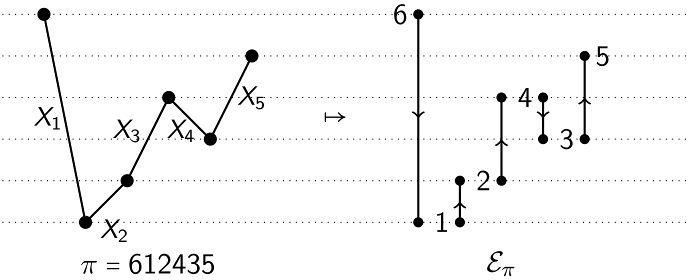


Encoding Patterns: Edge Diagrams



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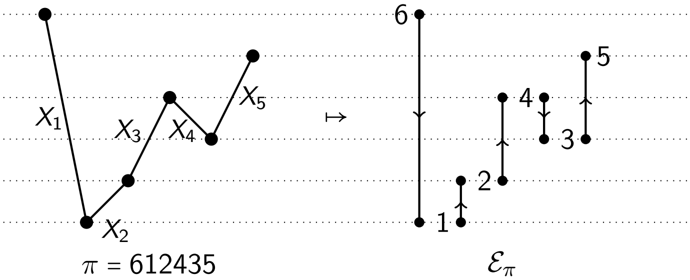

- Edge diagram, $\mathcal{E}_\pi$, is a set of directed edges corresponding to steps of π.

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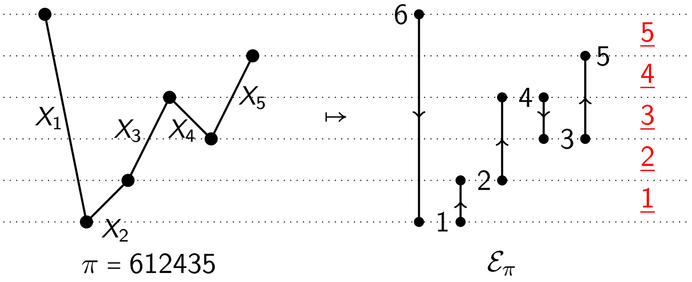

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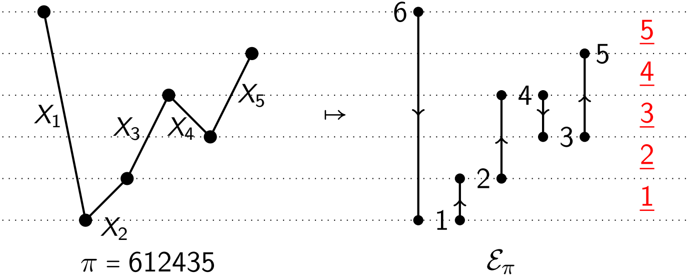
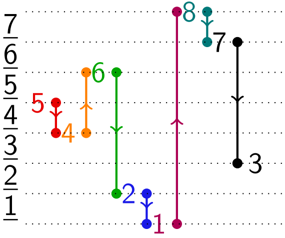


- Edge diagram, $\mathcal{E}_\pi$, is a set of directed edges corresponding to steps of π.
- Label **levels** as $\underline{1}, \underline{2}, \ldots, \underline{n-1}$.
- For $\mathcal{E}_\pi$, the edges form a "path."

Permuting the Levels of an Edge Diagram

- Apply permutation $\sigma \in S_{n-1}$ to the levels of an edge diagram.



$\mathcal{E}_\pi$

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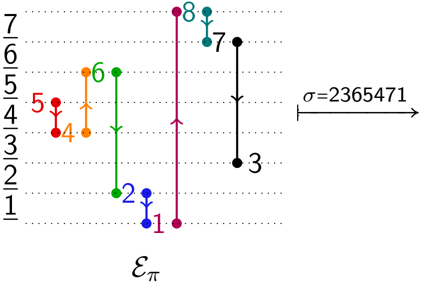

- Well-defined if edges "remain" edges

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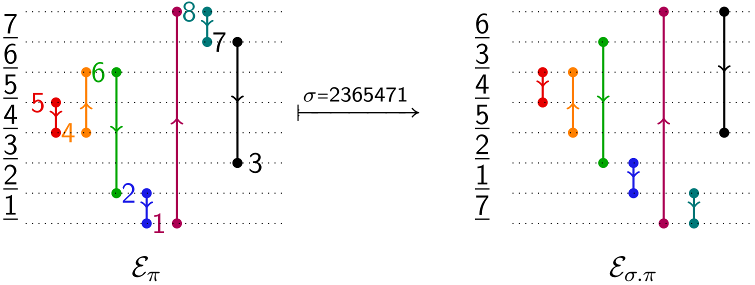

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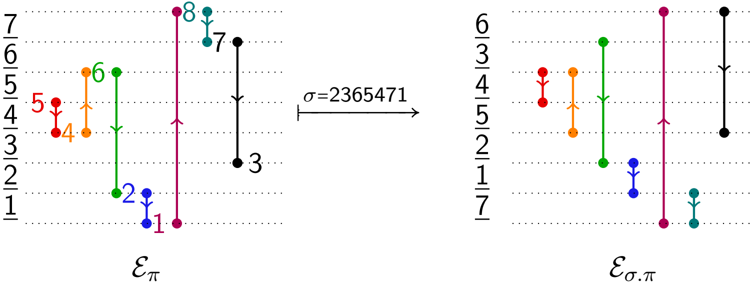

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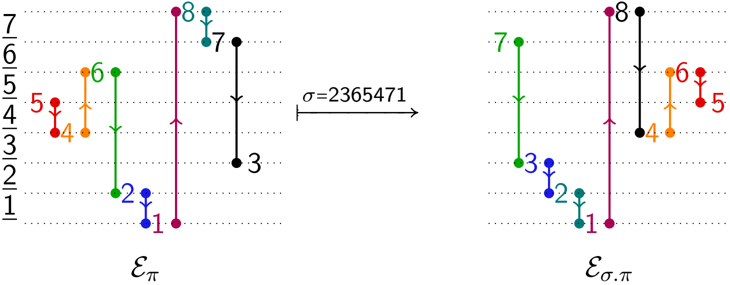

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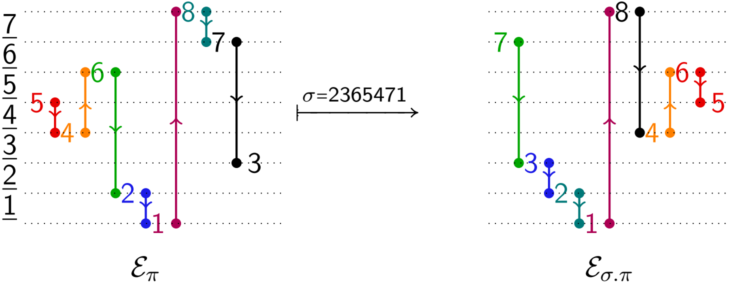

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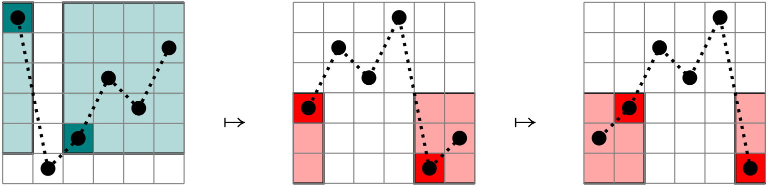
- Resulting diagram corresponds to a permutation if it forms a path

Lemma

$\pi \sim \tau$ if and only if there exists some $\sigma \in S_{n-1}$ such that $\mathcal{E}_{\sigma.\pi} = \mathcal{E}_\tau$.

Thank You!



$\pi = 612435 \mapsto 354612 \mapsto \tau = 235461$