A generalisation of two partition theorems of Andrews

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Outline

1. Basic facts about partitions and overpartitions
2. Schur's theorem and its generalisations
   - Schur's theorem
   - Schur's theorem for overpartitions
   - Andrews' generalisations of Schur's theorem
3. Generalisation of Andrews' theorems to overpartitions
   - The theorems
   - Idea of the proofs
   - Deducing $(r = 2, N = 3)$ from $(r = 1, N = 3)$ in the first theorem

Partitions

Definition

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Generating functions

Let n be a positive integer, H a set of positive integers, and let $p(n,H)$ denote the number of partitions of n whose parts lie in H. Then

$$1+\sum_{n\geq 1} p(n,H)q^n = \prod_{n\in H}\left(1+q^n+q^{2n}+\cdots\right) = \prod_{n\in H}\frac{1}{(1-q^n)}.$$

Let n be a positive integer, H a set of positive integers, and let $p'(n,H)$ denote the number of partitions of n into distinct parts whose parts lie in H. Then

$$1+\sum_{n\geq 1} p'(n,H)q^n = \prod_{n\in H}(1+q^n).$$

Overpartitions

Definition

Let n be a positive integer. An *overpartition* of n is a partition of n in which the first occurrence of a number may be overlined.

Example

There are 8 overpartitions of 3:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \text{ and } \overline{1}+1+1.$$

Generating function

An overpartition can be seen as a pair formed by a partition into distinct parts (the overlined parts) and a classical partition (the non-overlined parts). Thus the generating function for overpartitions is the following:

$$1 + \sum_{n \geq 1} \sum_{k \geq 0} \overline{p}(n,k,H) q^n d^k = \prod_{n \in H} \frac{1+q^n}{1-dq^n},$$

where $\overline{p}(n,k,H)$ denotes the number of overpartitions of n with k non-overlined parts, whose parts lie in H.

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Theorem (Schur 1926)

For any positive integer n, let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3 and $B(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_m$ of n such that

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3 & \text{if } \lambda_{i+1} \equiv 1, 2 \mod 3, \\ 4 & \text{if } \lambda_{i+1} \equiv 0 \mod 3. \end{cases}$$

Then $A(n) = B(n)$.

Example

The partitions counted by $A(10)$ are 10, $8+2$, $7+2+1$ and $5+4+1$.
The partitions counted by $B(10)$ are 10, $9+1$, $8+2$ and $7+3$.
There are 4 partitions in both cases.

Several proofs: Schur, Andrews, Bressoud, Bessenrodt, Alladi-Gordon, ...

Idea of Andrews' first proof

- $b_i(m,n)$: number of partitions $\lambda_1 + \cdots + \lambda_m$ of n counted by $B(n)$, having m parts, such that $\lambda_m \geq i$.

$$f_i(x) = f_i(x,q) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_i(m,n) x^m q^n.$$

- f_1 satisfies the following q-difference equation

$$f_1(x) = (1 + xq + xq^2) f_1(xq^3) + xq^3(1 - xq^3) f_1(xq^6).$$

- We solve it and obtain:

$$\sum_{n \geq 0} B(n) q^n = f_1(1) = \prod_{j=0}^{\infty} (1 + q^{3j+1})(1 + q^{3j+2}) = \sum_{n \geq 0} A(n) q^n.$$

Idea of Andrews' second proof

- $\pi_i(n)$: number of partitions $\lambda_1 + \cdots + \lambda_m$ of n counted by $B(n)$, such that $\lambda_1 \leq i$.

$$d_i = d_i(q) := 1 + \sum_{n=1}^{\infty} \pi_i(n) q^n.$$

- (d_i) satisfies the following recurrence equation

$$d_{3i+2} = \left(1 + q^{3i+1} + q^{3i+2}\right) d_{3i-1} + q^{3i}(1 - q^{3i}) d_{3i-4}.$$

- We solve it and obtain:

$$\sum_{n \geq 0} B(n) q^n = \lim_{i \to \infty} d_{3i+2} = \prod_{j=1}^{\infty} (1 + q^{3j-1})(1 + q^{3j-2}) = \sum_{n \geq 0} A(n) q^n.$$

Generalisations of Schur's theorem

Notation :

$$(a;q)_\infty = \prod_{k \geq 0} (1 - aq^k).$$

Schur

$$(-q;q^3)_\infty(-q^2;q^3)_\infty$$

Andrews

$$(-q;q^N)_\infty \cdots (-q^{2^{r-1}};q^N)_\infty$$

Lovejoy

$$\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(dq;q^3)_\infty(dq^2;q^3)_\infty}$$

D.

$$\frac{(-q;q^N)_\infty \cdots (-q^{2^{r-1}};q^N)_\infty}{(dq;q^N)_\infty \cdots (dq^{2^{r-1}};q^N)_\infty}$$

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Theorem (Lovejoy 2005)

Let $A(k,n)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3, with k non-overlined parts. Let $B(k,n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n, having k non-overlined parts and satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 1,2 \mod 3, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 0 \mod 3, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all k, n, $A(k,n) = B(k,n)$.

The case $k = 0$ corresponds to Schur's theorem.

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Notation

- In the following, r is a positive integer and $N \geq 2^r - 1$.
- $\beta_N(m)$:= the least positive residue of $m \mod N$.
- For $\alpha \in \{1, 2, \ldots, 2^r - 1\}$,
  $w(\alpha)$:= the number of powers of 2 appearing in the binary expansion of α,
  $v(\alpha)$:= the smallest power of 2 appearing in this expansion.

Andrews' first theorem

Theorem (Andrews)

Let $D(r, N; n)$ denote the number of partitions of n into distinct parts congruent to $2^0, 2^1, \ldots, 2^{r-1}$ modulo N.
Let $E(r, N; n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n into parts congruent to $1, 2, \ldots, 2^r - 1$ modulo N such that

$$\lambda_i - \lambda_{i+1} \geq N w(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all n, $D(r, N; n) = E(r, N; n)$.

Schur's theorem corresponds to the case $r = 2$, $N = 3$.

Andrews' second theorem

Theorem (Andrews)

Let $F(r, N; n)$ denote the number of partitions of n into distinct parts congruent to $-2^0, -2^1, \dots, -2^{r-1}$ modulo N.
Let $G(r, N; n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n into parts congruent to $-1, -2, \dots, -2^r + 1$ modulo N such that

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and $\lambda_s \geq N\left(w(\beta_N(-\lambda_s) - 1\right).$
Then for all n, $F(r, N; n) = G(r, N; n)$.

Again, Schur's theorem corresponds to the case $r = 2$, $N = 3$. But for other values, the two theorems are different.

The case $r = 3, N = 7$

Theorem (Andrews 1)

Let $A(n)$ denote the number of partitions of n into distinct parts congruent to 1,2 *or* 4 *modulo* 7*. Let $B(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$, where*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 7 \text{ if } \lambda_{i+1} \equiv 1,2,4 \mod 7, \\ 12 \text{ if } \lambda_{i+1} \equiv 3 \mod 7, \\ 10 \text{ if } \lambda_{i+1} \equiv 5,6 \mod 7, \\ 15 \text{ if } \lambda_{i+1} \equiv 0 \mod 7. \end{cases}$$

Then for all n, $A(n) = B(n)$.

The case $r = 3, N = 7$

Theorem (Andrews 2)

Let $C(n)$ denote the number of partitions of n into distinct parts congruent to 3,5 or 6 modulo 7. Let $D(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$, where

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 7 \text{ if } \lambda_i \equiv 3,5,6 \mod 7, \\ 12 \text{ if } \lambda_i \equiv 4 \mod 7, \\ 10 \text{ if } \lambda_i \equiv 1,2 \mod 7, \\ 15 \text{ if } \lambda_i \equiv 0 \mod 7, \end{cases}$$

and $\lambda_s \neq 1,2,4,7$.
Then for all n, $C(n) = D(n)$.

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The first theorem

Theorem (D. 2014)

Let $D(r, N; k, n)$ denote the number of overpartitions of n into parts congruent to $2^0, 2^1, \dots, 2^{r-1}$ modulo N, having k non-overlined parts. Let $E(r, N; k, n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n into parts congruent to $1, 2, \dots, 2^r - 1$ modulo N, having k non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq N\left(w\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}),$$

Then for all $k, n \geq 0$, $D(r, N; k, n) = E(r, N; k, n)$.

The case $k = 0$ corresponds to Andrews' first theorem.
The case $N = 3$, $r = 2$ corresponds to Schur's theorem for overpartitions.

The second theorem

Theorem (D. 2014)

Let $F(r,N;k,n)$ denote the number of overpartitions of n into parts congruent to $-2^0,-2^1,\dots,-2^{r-1}$ modulo N, having k non-overlined parts.
Let $G(r,N;k,n)$ denote the number of overpartitions $\lambda_1+\cdots+\lambda_s$ of n into parts congruent to $-1,-2,\dots,-2^r+1$ modulo N, having k non-overlined parts, such that

$$\lambda_i-\lambda_{i+1}\geq N\left(w\left(\beta_N(-\lambda_i)\right)-1+\chi(\overline{\lambda_{i+1}})\right)+v(\beta_N(-\lambda_i))-\beta_N(-\lambda_i),$$

$$\lambda_s\geq N\left(w(\beta_N(-\lambda_s))-1\right),$$

Then for all $k,n\geq 0$, $F(r,N;k,n)=G(r,N;k,n)$.

The case $k=0$ corresponds to Andrews' second theorem.
Again, the case $N=3$, $r=2$ corresponds to Lovejoy's theorem.

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Proof of the first theorem

- Define $b_i^r(k,m,n)$ as the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ counted by $E(r,N;k,n)$, having m parts, such that $\lambda_m \geq i$, and

$$f_i^r(x) = f_i^r(x,q,d) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} b_i^r(k,m,n) x^m d^k q^n.$$

- Find the q-difference equation $(eq_{r,N})$ satisfied by $f_1^r(x)$.
- Show by induction on r that a function f satisfying $(eq_{r,N})$ and $f(0) = 1$ satisfies

$$f(1) = \prod_{k=0}^{r-1} \frac{(-q^{2^k};q^N)_\infty}{(dq^{2^k};q^N)_\infty},$$

which is the generating function for overpartitions counted by $D(r,N;k,n)$.

Proof of the second theorem

- Define $\psi_i^r(k,n)$ as the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ counted by $G(r,N;k,n)$, such that $\lambda_1 \leq i$, and

$$g_i^r = g_i^r(q,d) := 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \psi_i^r(k,n) q^n d^k.$$

- Find the recurrence equation $(rec_{r,N})$ satisfied by $(g_i^r)_{r\in\mathbb{N}}$.
- Show by induction on r that the limit as $i \to \infty$ of a sequence (g_i) satisfying $(rec_{r,N})$ and $g_0 = 1$ equals

$$\prod_{j=0}^{r-1} \frac{(-q^{N-2^k};q^N)_\infty}{(dq^{N-2^k};q^N)_\infty},$$

which is the generating function for overpartitions counted by $F(r,N;k,n)$.

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The theorem for $r = 1, N = 3$

Theorem

Let $A(k,n)$ denote the number of overpartitions of n into parts congruent to 1 modulo 3, having k non-overlined parts. Let $B(k,n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_m$ of n into parts congruent to 1 modulo 3, having k non-overlined parts, such that

$$\lambda_i - \lambda_{i+1} \geq 0 + 3\chi(\overline{\lambda_{i+1}}).$$

Then $A(k,n) = B(k,n)$.

We have

$$\begin{aligned} b_1^1(k,m,n) - b_4^1(k,m,n) &= b_1^1(k-1,m-1,n-1) \\ &\quad + b_1^1(k,m-1,n-3(m-1)-1). \end{aligned}$$

Thus

$$b_1^1(k,m,n) = b_1^1(k,m,n-3m) + b_1^1(k-1,m-1,n-1) \\ + b_1^1(k,m-1,n-3m+2),$$

and

$$(1-dxq)f_1^1(x) = (1+xq)f_1^1(xq^3).$$

Iterating gives

$$f_1^1(x) = \prod_{k\geq 0} \frac{(1+xq^{3k+1})}{(1-dxq^{3k+1})} f_1^1(0) = \prod_{k\geq 0} \frac{(1+xq^{3k+1})}{(1-dxq^{3k+1})}.$$

So

$$f_1^1(1) = \prod_{k\geq 0} \frac{(1+q^{3k+1})}{(1-dq^{3k+1})}.$$

Deducing Lovejoy's theorem

In the same way as before, we obtain a q-difference equation:

$$(1-dxq)(1-dxq^2)f_1^2(x) = (1+xq+xq^2+dxq^3-dx^2q^3-dx^2q^6)f_1^2(xq^3) + xq^3(1-xq^3)f_1^2(xq^6),$$

and $f_1^2(0)=1$.
Let

$$F(x) := f_1^2(x)\prod_{k=0}^{\infty}\frac{1-dxq^{3k+2}}{1-xq^{3k}}.$$

Then $F(0)=1$ and

$$(1-dxq)(1-x)F(x) = (1+xq+xq^2+dxq^3-dx^2q^3-dx^2q^6)F(xq^3) + xq^3(1-dxq^5)F(xq^6).$$

Deducing Lovejoy's theorem

Let

$$F(x) =: \sum_{n\geq 0} A_n x^n.$$

Then $A_0 = 1$ and

$$(1-q^{3n})A_n = (1+dq+q^{3n-2})(1+q^{3n-1})A_{n-1} - dq(1+q^{3n-1})(1+q^{3n-4})A_{n-2}.$$

Now

$$a_n := \frac{A_n}{\prod_{k=0}^{n-1}(1+q^{3k+2})}.$$

Then $a_0 = 1$ and

$$(1-q^{3n})a_n = (1+dq+q^{3n-2})a_{n-1} - dqa_{n-2}.$$

Deducing Lovejoy's theorem

Let

$$G(x) := \sum_{n \geq 0} a_n x^n.$$

Then $G(0) = 1$ and

$$(1-x)(1-dxq)G(x) = (1+xq)G(xq^3).$$

Now

$$g(x) := G(x) \prod_{k \geq 0} (1 - xq^{3k}).$$

Then $g(0) = 1$ and

$$(1-dxq)g(x) = (1+xq)g(xq^3).$$

This is $(eq_{1,3})$, so

$$g(1) = f_1^1(1) = \prod_{k \geq 0} \frac{(1+q^{3k+1})}{(1-dq^{3k+1})}.$$

Deducing Lovejoy's theorem

Theorem (Appell's Lemma)

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\lim_{n\to\infty} a_n$ *is finite. Then*

$$\lim_{x\to 1}(1-x)\sum_{n\geq 0} a_n x^n = \lim_{n\to\infty} a_n.$$

Then

$$\begin{aligned}\frac{g(1)}{\prod_{k\geq 1}(1-q^{3k})} &= \lim_{x\to 1}(1-x)G(x)\\ &= \lim_{x\to 1}(1-x)\sum_{n\geq 0} a_n x^n\\ &= \lim_{n\to\infty} a_n.\end{aligned}$$

Deducing Lovejoy's theorem

Thus

$$\lim_{n\to\infty} a_n = \prod_{k\geq 0} \frac{(1+q^{3k+1})}{(1-dq^{3k+1})(1-q^{3k+3})},$$

so

$$\lim_{n\to\infty} A_n = \prod_{k\geq 0} \frac{(1+q^{3k+1})(1+q^{3k+2})}{(1-dq^{3k+1})(1-q^{3k+3})},$$

and

$$\begin{aligned}\lim_{x\to 1}(1-x)F(x) &= \lim_{n\to\infty} A_n \\ &= f_1^2(1) \prod_{k\geq 0} \frac{1-dq^{3k+2}}{1-q^{3k+3}}.\end{aligned}$$

Thus

$$f_1^2(1) = \prod_{k\geq 0} \frac{(1+q^{3k+1})(1+q^{3k+2})}{(1-dq^{3k+1})(1-dq^{3k+2})}. \quad \square$$

The q-difference equation in the general case

$$\prod_{j=0}^{r-1}\left(1-dxq^{2^j}\right)f_1^r(x)=f_1^r(xq^N)$$
$$+\sum_{j=1}^{r}\left(\sum_{m=0}^{r-j}d^m\sum_{\substack{\alpha<2^r\\ w(\alpha)=j+m}}xq^{\alpha}\left((-x)^{m-1}\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}\right.\right.$$
$$\left.\left.+(-x)^m\begin{bmatrix}j+m\\ m\end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right)f_1^r\left(xq^{jN}\right).$$

Perspectives

- Bijective proofs of the new theorems?
- Generalise other Rogers-Ramanujan type partition identities to overpartitions (Capparelli,...)
- Oh[1] showed that Andrews' theorems are linked to quantum algebra. Is it also the case for their generalisations?

[1] S. Oh. "The Andrews-Olsson identity and Bessenrodt insertion algorithm on Young walls". In: *Eur. J. Comb.* 43 (2015), pp. 8–31.