Enumeration with Moon Polyominoes and Beyond

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Outline

1. The Motivation

Permutation Statistics

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation,

- $\mathrm{Inv}(\sigma) = \{(a_i, a_j) : \; i < j, \; a_i > a_j\}$, $\quad$ $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$.
  and $\mathrm{coinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$.

- $\mathrm{Des}(\sigma) = \{i \; : \; a_i > a_{i+1}, 1 \le i < n\}$, $\quad$ $\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|$.

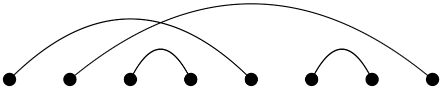
- $\mathrm{maj}(\sigma) = \displaystyle\sum_{i \in \mathrm{DES}(\sigma)} i = \sum_i \mathrm{des}(a_i a_{i+1} \cdots a_n)$.

Permutation Statistics

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation, e.g. $\sigma = 3712645$

- $\mathrm{Inv}(\sigma) = \{(a_i, a_j) : \ i < j, \ a_i > a_j\}$, $\quad \mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$.
  and $\mathrm{coinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$.
  $\mathrm{Inv}(\sigma) = \{31, 32, 71, 72, 76, 74, 75, 64, 65\}$, $\mathrm{inv}(\sigma) = 9$,
  $\mathrm{coinv}(\sigma) = 12$
- $\mathrm{Des}(\sigma) = \{i \ : \ a_i > a_{i+1}, 1 \leq i < n\}$, $\quad \mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|$.
  $\mathrm{Des}(\sigma) = \{2, 5\}$, $\mathrm{des}(\sigma) = 2$.
- $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{DES}(\sigma)} i = \sum_i \mathrm{des}(a_i a_{i+1} \cdots a_n)$.
  $\mathrm{maj}(\sigma) = 7$

Matchings: crossings and nestings



$$\mathrm{cr}_2(\alpha) = 1, \qquad \mathrm{ne}_2(\alpha) = 3.$$

Let

$$L_n(p,q) = \sum_{\alpha \in M(2n)} p^{\mathrm{cr}_2(\alpha)} q^{\mathrm{ne}_2(\alpha)},$$

Theorem (Touchard, Riordan, Kasraoui & Zeng)

$$\sum_{n \geq 0} L_n(p,q) z^n = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{[2]_{p,q} z}{1 - \cfrac{[3]_{p,q} z}{\cdots}}}}$$

If we fix the sets of minimal elements and maximal elements,

$$\sum_{M} p^{\mathrm{cr}_2(M)} q^{\mathrm{ne}_2(M)} = (p+q)(p+q).$$

In general, it is always a product of (p,q)-integers, where $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$.

Problem 1.

A crossing/nesting with edges (i_1, j_1), (i_2, j_2) is:

$$\begin{cases} \text{odd} & \text{if } i_1 \text{ is odd} \\ \text{even} & \text{if } i_1 \text{ is even} \end{cases}$$

Observation

$$\sum_M p^{\text{odd cr}_2 + \text{even ne}_2} q^{\text{even cr}_2 + \text{odd ne}_2} = \sum_M p^{\text{cr}_2(M)} q^{\text{ne}_2(M)}$$

Problem 2.

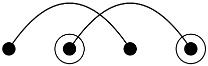
A crossing with edges (i_1, j_1), (i_2, j_2) is:

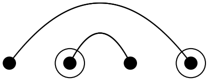
$$\begin{cases} \text{small} & \text{if } i_2 + j_2 \leq 2n; \\ \text{large} & \text{otherwise} \end{cases}$$

A nesting with edges (i_1, j_1), (i_2, j_2) is

$$\begin{cases} \text{small} & \text{if } i_2 + j_1 \leq 2n; \\ \text{large} & \text{otherwise} \end{cases}$$

Back



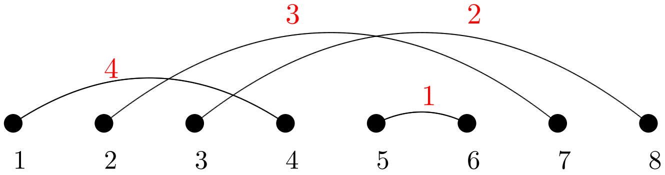


Observation

$$\sum_M p^{\text{small cr}_2 + \text{large ne}_2} q^{\text{large cr}_2 + \text{small ne}_2} = \sum_M p^{\text{cr}_2(M)} q^{\text{ne}_2(M)}$$

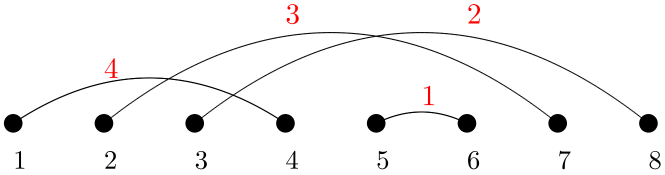
Problem 3.

A major index for matching:



Problem 3.

A major index for matching:



sequence σ_i 432 132 32 2

$$\mathrm{pmaj}(M) := \sum_i \mathrm{des}(\sigma_i).$$

[Chen, Gessel, Y & Yang 2008]: pmaj has the same distribution as cr_2.

Question: Foata-type transformation?

2. A Combinatorial Model

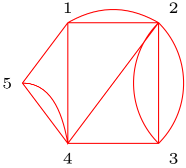
Fillings of polyominoes

Polyomino: a finite subset of $\mathbb{Z}^2$, represented by square cells; each cell is assigned a natural number.

- 01-fillings of rectangles: Permutations and Words
- Triangular shape: graphs on $[n]$

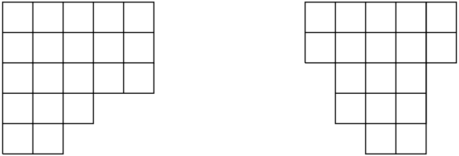
e.g. $\sigma = 4132$

	1	2	3	4
1				1
2	1			
3			1	
4		1		



	5	4	3	2
1	1	1	0	2
2	0	1	3	
3	0	1		
4	2			

- Ferrers diagrams (Backelin, West & Xin; Krattenthaler, de Mier): Graphs with given degree sequence, matchings, set partitions with given MIN/ MAX block-elements, rook placements
- Stack polyominoes (Jonsson; Jonsson & Welker): pattern avoidance of set partitions (Jelínek & Mansour)

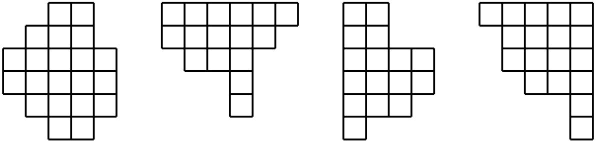


Moon polyominoes

convex any column or row is connected.

intersection-free Every two rows are comparable, i.e., the column-coordinates of the longer one cover those of the shorter one.

moon polyomino (L-connected) a convex and intersection-free polyomino (Rubey, Kasraoui,...)



We consider 01-fillings.

A combinatorial hierarchy

permutation/word ⊆ matching/set partitions ⊆ graphs
⊆ Ferrers diagrams ⊆ stack polyominoes
⊆ moon polyominoes
⊆ more general polyominoes

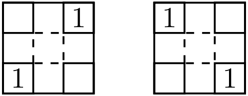
Allow various approaches and techniques, e.g.

- fix the polyomino and change the fillings
- transform the polyominoes
- bijection, induction,
- tableaux operations ...

3. Combinatorial Statistics

Inversion: basic correspondences

	inversion/coinversion	in	permutations
$\Longleftrightarrow$	crossings/nestings of two edges	in	matchings
$\Longleftrightarrow$	northeast/southeast chains of size 2	in	01-fillings



Denote by $\mathrm{ne}_2(M)$ and $\mathrm{se}_2(M)$ the numbers of ne-/se-chains of size 2 in a filling M.

A nice theorem

Given $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $\mathbf{e} = (e_1, \dots, e_m) \in \{0,1\}^m$. Let $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ be the set of 01-fillings of $\mathcal{M}$ with row-sum $\mathbf{s}$ and column-sum $\mathbf{e}$.

Theorem (Kasraoui 2010)

$$\sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} q^{\mathrm{ne}_2(M)} p^{\mathrm{se}_2(M)} = \prod_{i=1}^{n} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_{p,q}$$

Remark

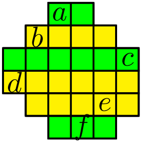
1. Not true if allow arbitrary row sum *and* arbitrary column sum
2. Not true if not *convex* or *intersection-free*

It contains results on

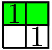
- permutations: $\sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{inv}(\pi)} q^{\mathrm{coinv}(\pi)} = [n]_{p,q}!$.
- matchings [de Sainte-Catherine'83]
- set partitions [Kasraoui & Zeng'06]
- crossings and alignments for permutation [Corteel'07]
- linked partitions [Chen, Wu & Y'08]

A mixed variant

Bicolor the rows of $\mathcal{M}$ and mix the 2-chains by the position of the top cell.



Define top-mixed statistics

$\alpha(M)$:  or 

$\beta(M)$:  or 

$\alpha(M) = 3$ with chains ef, ae, af;
$\beta(M) = 4$ with chains be, bf, cd, ce.

Over $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$

Theorem (Chen, Wang, Y & Zhao 2010)

The joint distribution $(\alpha(M), \beta(M))$ is always symmetric and independent of the bi-coloring. In particular,

$$\sum_{M} p^{\alpha(M)} q^{\beta(M)} = \sum_{M} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)}$$

Also true if one mixes by the bottom cell, or bi-coloring columns of $\mathcal{M}$.
This explains Problem 1. ◂ Problem 1

Charged polyominoes

Equip with $\mathcal{M}$ a charge function $C : \mathcal{M} \to \{\pm 1\}$.
For a 2×2 submatrix S of $\mathcal{M}$, set $\mathrm{sgn}(S)$ as the charge of its lower-right corner.

Definition

A chain with the support matrix S is ***positive*** *with respect to C if it is*

1. *a northeast chain with* $\mathrm{sgn}(S) = 1$, *or*
2. *a southeast chain with* $\mathrm{sgn}(S) = -1$.

Otherwise, the chain is negative.

(a) Positive chains (b) Negative chains

Conjecture

Then the distribution of $(\mathrm{pos}_C(M), \mathrm{neg}_C(M))$ *does not depend on the charge function* C. *Consequently,*

$$\sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\mathrm{pos}_C(M)} q^{\mathrm{neg}_C(M)} = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)}.$$

Not always!

Theorem (Wang & Y 2013)

The conjecture is true if the polyomino is top aligned or left aligned.

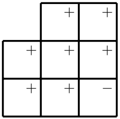
This explains Problem 2.

Not always!

Theorem (Wang & Y 2013)

The conjecture is true if the polyomino is top aligned or left aligned.

This explains Problem 2.
In general,

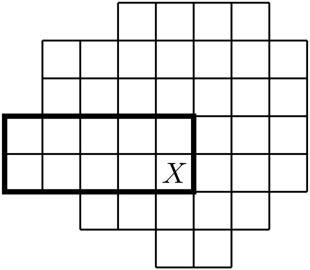


$$\sum_M p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = p^2 + 2pq + q^2,$$

$$\sum_M p^{\mathrm{pos}_C(M)} q^{\mathrm{neg}_C(M)} = 2p^2 + 2q^2.$$

The restrictive positivity

For a cell $X \in \mathcal{M}$, define $\mathcal{R}_X$, *the box of* X, to be the widest rectangle contained in $\mathcal{M}$ whose lower right corner is X.



A 2×2 submatrix is *restrictive* if it is contained in the box of its lower right corner.

Definition

Let M be a 01-filling of a charged moon polyomino $\mathcal{M}$ with a charge function C. A chain with the support matrix S is ***restrictively positive*** *with respect to C if it is*

1. *a northeast chain with* $\mathrm{sgn}(S) = 1$ ***or not restrictive,***
2. *a southeast chain with* $\mathrm{sgn}(S) = -1$ ***and restrictive.***

Otherwise, the chain is restrictively negative.

Symmetry in the general case

Let $\overline{\text{pos}}_C(M)$ and $\overline{\text{neg}}_C(M)$ be the numbers of restrictively positive chains and restrictively negative chains of M with respect to C. Set

$$\overline{F}_C(p,q) = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\overline{\text{pos}}_C(M)} q^{\overline{\text{neg}}_C(M)}.$$

Theorem (Wang & Y 2013)

The bi-variate generating function $\overline{F}_C(p,q)$ does not depend on the charge function C. Consequently,

$$\overline{F}_C(p,q) = \overline{F}_+(p,q) = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\text{ne}_2(M)} q^{\text{se}_2(M)}.$$

Idea of the proofs:

1. In a rectangle shape, establish a bijection between fillings while changing the color of one row, or the charge of one cell.
2. In general, gradually change the bi-coloring to a single coloring, or the charge function to *all positive*.

The major index

Goal: A statistic that can be defined in a uniform way on permutations / matchings / set partitions, which specializes to the major index on permutations.

Theorem (Chen, Poznanović, Y & Yang 2010)

The major index can be extended to 01*-fillings in* $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$*. It has the same distribution as* ne_2*.*

Definition of the major index

Let $\mathcal{M}$ be a top-aligned stack polyomino. Let M be a filing of $\mathcal{M}$ with at most one 1 in each row.

- If $\mathcal{M}$ is a rectangle,

  des(M) := #ne-chains formed by 1s in adjacent nonempty rows.

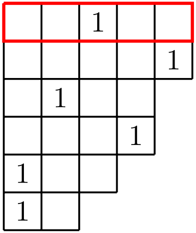
- Let $M(r_i)$ be the maximal rectangle with bottom row r_i.
- Define

$$\mathrm{maj}(M) := \sum_{i=1}^{n} \mathrm{des}(M(r_i)).$$

An example

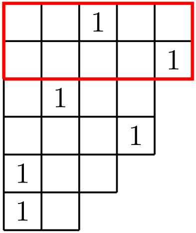
		1		
				1
	1			
			1	
1				
1				

An example



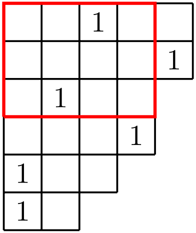
- $\mathrm{des}(M(r_1)) = 0.$

An example



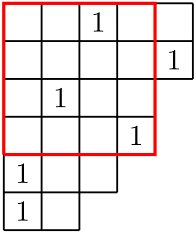
- $\text{des}(M(r_1)) = 0.$
- $\text{des}(M(r_2)) = 0.$

An example



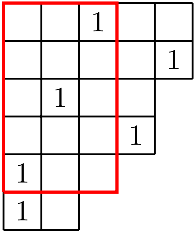
- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$

An example



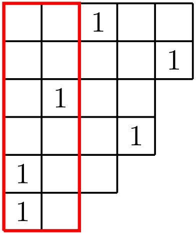
- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$
- $\mathrm{des}(M(r_4)) = 1.$

An example



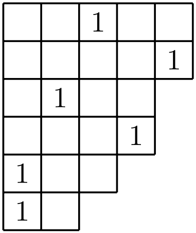
- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$
- $\mathrm{des}(M(r_4)) = 1.$
- $\mathrm{des}(M(r_5)) = 2.$

An example



- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$
- $\mathrm{des}(M(r_4)) = 1.$
- $\mathrm{des}(M(r_5)) = 2.$
- $\mathrm{des}(M(r_6)) = 1.$

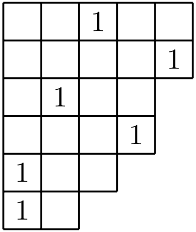
An example



- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$
- $\mathrm{des}(M(r_4)) = 1.$
- $\mathrm{des}(M(r_5)) = 2.$
- $\mathrm{des}(M(r_6)) = 1.$

$\mathrm{maj}(M)=5.$

An example



- $\mathrm{des}(M(r_1)) = 0.$
- $\mathrm{des}(M(r_2)) = 0.$
- $\mathrm{des}(M(r_3)) = 1.$
- $\mathrm{des}(M(r_4)) = 1.$
- $\mathrm{des}(M(r_5)) = 2.$
- $\mathrm{des}(M(r_6)) = 1.$

$\mathrm{maj}(M)=5.$

In general, $\mathrm{maj}(M)$ is defined as an alternating sum.

$$\mathrm{maj}(M) := \sum_i \mathrm{maj}(R_i) - \sum_i \mathrm{maj}(R_i \cap R_{i+1})$$

where $R_1, \ldots R_k$ are maximal rectangles.

A Foata-type bijection

Theorem

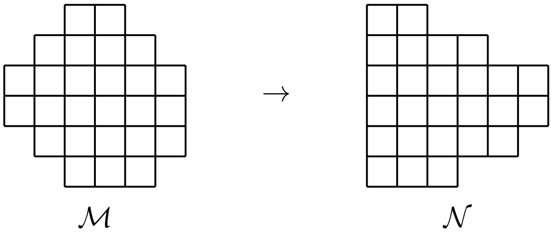
$\text{maj}(M)$ *has the same distribution as* $\text{ne}_2(M)$.

Define a bijection $\phi : \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s}) \to \mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ recursively such that $\text{ne}_2(\phi(M)) = \text{maj}(M)$.

1. If $\mathcal{M}$ has only one row, $\phi(M) = M$.
2. If $\mathcal{M}$ has more than one row
   1. M_1 = remove the 1st row from $\mathcal{M}$.
   2. Inductive step $N_1 = \phi(M_1)$
   3. **Modify N_1 to get N_2**
   4. $\phi(M)$ = add the 1st row back to N_2.

[CPYY] Foata-type transformation can be defined on fillings of **left-aligned stack polyominoes** by a set of row operations.

For general polyominoes: move shorter columns to the right.



- Construct a bijection from fillings of $\mathcal{M}$ to $\mathcal{N}$ that preserves the major index.
- Construct a bijection from fillings of $\mathcal{M}$ to $\mathcal{N}$ that preserves ne_2.

(This answers problem 3.)

A little results on descent

Recall on permutations: fix D $= \{d_1, d_2, \ldots, d_k\} \subset [n-1]$ and let

$$\beta(D,q) = \sum_{\sigma:\mathrm{Des}(\sigma)=D} q^{\mathrm{inv}(\sigma)}, \qquad \alpha(D,q) = \sum_{\sigma:\mathrm{Des}(\sigma)\subseteq D} q^{\mathrm{inv}(\sigma)}$$

Then

Theorem

$$\alpha(D;q) = \begin{bmatrix} n \\ d_1, d_2-d_1, \ldots, n-d_k \end{bmatrix}_q,$$

and

$$\beta(D;q) = \det \left[\begin{bmatrix} n-d_i \\ d_{j+1}-d_i \end{bmatrix}_q \right].$$

Restricted fillings in Ferrers diagram

Let $\mathcal{F}$ be a Ferrers diagram with row lengths $r_1 \leq r_2 \leq \cdots \leq r_n$.

Theorem (Song & Y 2012)

$$\alpha_F(D,q) = \prod_{i=0}^{k} LP(\vec{s}_i; q), \qquad \beta_F(D,q) = \det[f_{i,j+1}]$$

where $f(i,j) = LP(\vec{s}_{i+1}, \ldots, \vec{s}_j; q)$ *for* $i \leq j$.

Here $s_i = r_i - i + 1$, and $LP(\vec{s}; q)$ is the area-enumerator of lattice paths with right boundary $\vec{s}$.

- an extension of Stanley's inv q-analog of G.F. of Eulerian polynomials
- a new G.F. of descent polynomials for permutations with maximal drop size, [Chung, Claesson, Dukes, Graham'10].

4. More General Shapes?

What happens if one swaps the rows of the polyomino?

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Idea: extend to polyominoes that allow arbitrary permutations of rows.

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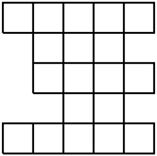
Idea: extend to polyominoes that allow arbitrary permutations of rows.

Layer Polyomino: intersection-free and row-convex, but not necessarily column-convex.

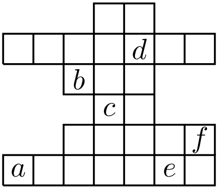


Regular chains in fillings of layer polyomino

A 2×2 submatrix

$$S = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2) \in \mathcal{L} : i_1 < i_2, j_1 < j_2\}.$$

ne-chain: S with $(i_1, j_2), (i_2, j_1)$ filled with 1.
se-chain: S with $(i_1, j_1), (i_2, j_2)$ filled with 1.



4 ne-chains: bd, cd, ad, ef 2 se-chains: df, de

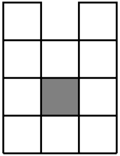
Main result on Layer Polyomino $\mathcal{L}$

Fix row sum $\mathbf{s}$ and column sum $\mathbf{e}$.

Theorem (Phillipson, Y & Yeh 2013, 2015)

1. *If either* $\mathbf{s}$ *or* $\mathbf{e}$ *is a 01-vector, then permuting rows of* $\mathcal{L}$ *does not change the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$. *Consequently, the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$ *is symmetric.*
2. *For arbitrary* $\mathbf{s}, \mathbf{e}$, *the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$ *may not be symmetric.*

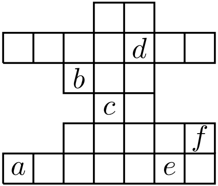
Cannot get rid of all convexity.



Strong Chains

Definition

An ne/se-chain is ***strong*** *if the minimal rectangle containing it is also in $\mathcal{L}$.*



3 strong ne-chains: bd, cd, ef, no strong se-chains.

Let $\mathrm{ne}_2^{\square}$ and $\mathrm{se}_2^{\square}$ be the number of strong ne-and se-chains of length 2. Then

Theorem (Phillipson, Y & Yeh 2013, 2015)

1. $(\mathrm{ne}_2^{\square}, \mathrm{se}_2^{\square})$ *is not preserved under permutations of rows*
2. *The distribution of* $(\mathrm{ne}_2^{\square}, \mathrm{se}_2^{\square})$ *is symmetric if either row-sum or column-sum is a* 01*-vector.*
3. *An involution for (2) is constructed.*
4. *Fix both row sum and column sum in* $\mathbb{N}$*, the distribution of* $\mathrm{ne}_2^{\square}$ *and* $\mathrm{se}_2^{\square}$ *may not be the same; however*
5. *there is a bijection between fillings with no strong ne-chains to fillings with no strong se-chains, for both* 01*-fillings and* $\mathbf{N}$*-fillings.*

How about $\mathrm{ne}(M)$, the size of maximal northeast chains?

Exchanging rows WILL change the distribution of this statistic!

One needs more restrictions on layer polyomino.
Please come to the talk at 12:00.
[Poznanović & Yan]:

Maximal increasing sequences in fillings of almost-moon polyominoes

Acknowledgments

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