Dual Filtered Graphs

Becky Patrias

(with Pasha Pylyavskyy)

FPSAC 2015

July 7, 2015

Outline

- Dual graded graphs
- Dual filtered graphs
  - Trivial construction
  - Pieri construction
  - Möbius construction
  - Möbius via Pieri phenomenon

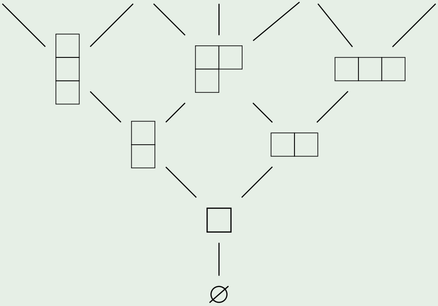
Dual graded graphs

A graded graph is a triple $G = (P, \rho, E)$, where

- P is a countable set of vertices,
- $\rho : P \to \mathbb{Z}$ is a rank function,
- and E is a multiset of edges/arcs (x, y), where $\rho(y) = \rho(x) + 1$.

Example

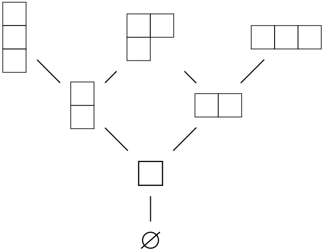
Young's lattice



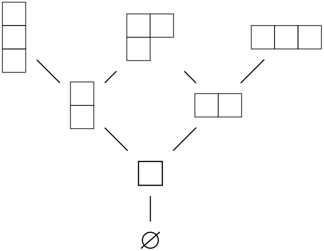
Dual graded graphs

To make a dual graded graph:

- Take two graded graphs with a **common vertex set and rank function**, $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$.



G_1

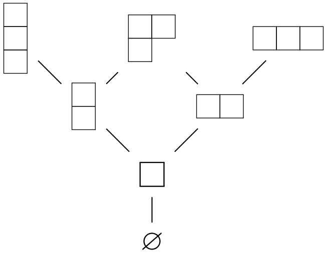


G_2

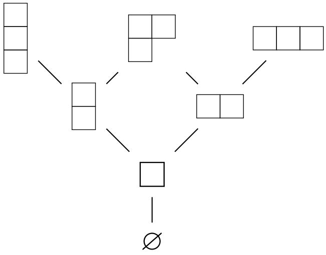
Dual graded graphs

To make a dual graded graph:

- Take two graded graphs with a **common vertex set and rank function**, $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$.
- Define an **oriented graded graph** $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank.



G_1

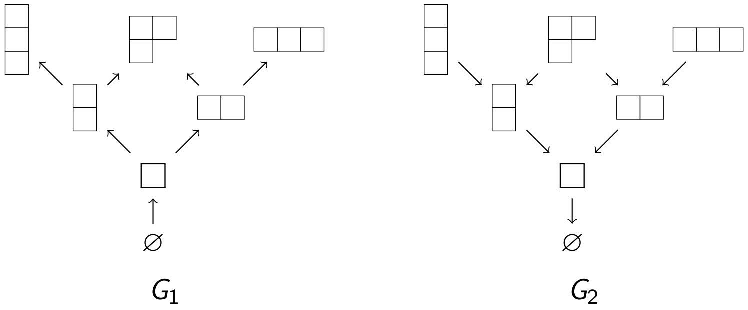


G_2

Dual graded graphs

To make a dual graded graph:

- Take two graded graphs with a **common vertex set and rank function**, $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$.
- Define an **oriented graded graph** $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank.

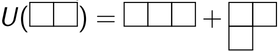


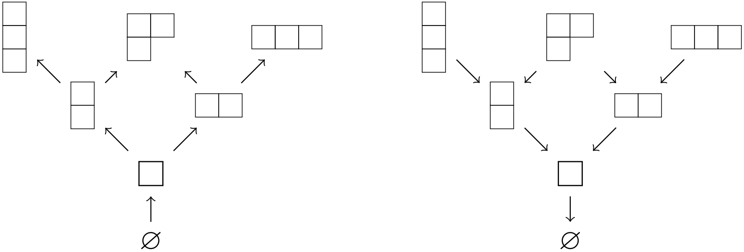
Dual graded graphs

For any field K of characteristic zero, the formal linear combinations of vertices of G form the vector space KP.

We next define the *up* and *down* operators $U, D \in End(KP)$.

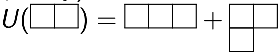
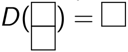
For any x in G,

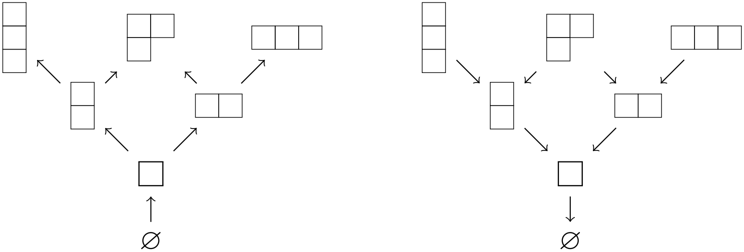
- let Ux be the sum of elements of G directly above x (with multiplicity), and
  - 



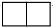
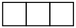
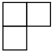
Dual graded graphs

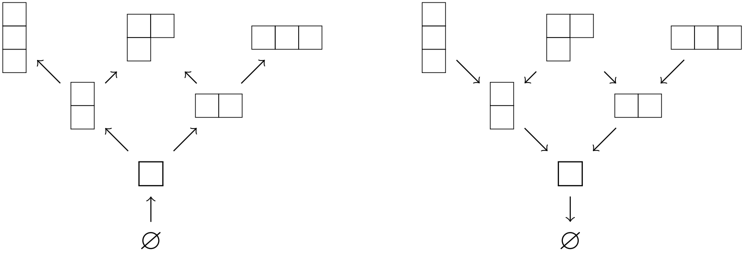
For any x in G,

- let Ux be the sum of elements of G directly above x (with multiplicity), and
  - 
- let Dx be the sum of elements of G directly below x (with multiplicity).
  - 



For any x in G,

- let Ux be the sum of elements of G directly above x (with multiplicity), and
  - $U($$) =$  $+$ 
- let Dx be the sum of elements of G directly below x (with multiplicity).
  - $D($$) =$ 
  - $D(\varnothing) = 0$



Dual graded graphs

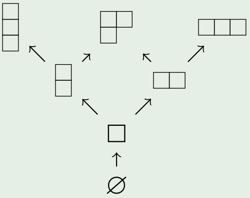
Definition (Fomin)

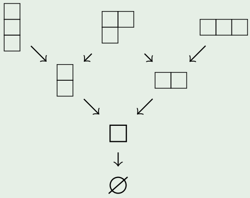
We call G_1 and G_2 **dual graded graphs** if for any x in G,

$$(DU - UD)x = x.$$

Example

Young's lattice is a self-dual graded graph (or differential poset (Stanley)).



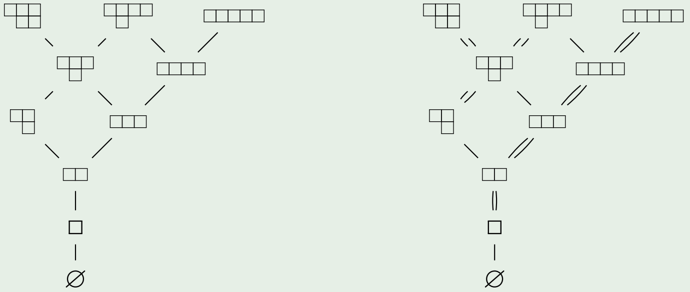


$$(DU - UD)(\ydiagram{2}) = D(\ydiagram{3} + \ydiagram{2,1}) - U(\ydiagram{1}) = (\ydiagram{2} + \ydiagram{1,1} + \ydiagram{2}) - (\ydiagram{1,1} + \ydiagram{2}) = \ydiagram{2}$$

Example of dual graded graphs

Example

Shifted Young's lattice forms a dual graded graph.



Why dual graded graphs?

- Invented as a tool to study the Robinson-Schensted insertion algorithm.
- Significant in areas where the correspondence appears, such as Schubert calculus and the study of towers of representations of algebras.

Dual graded graphs and Robinson-Schensted

The Robinson-Schensted algorithm gives a bijection between permutations of n and a pair (P, Q) of standard Young tableaux of the same shape with n boxes.

Example

Take $w = 2413 \in \mathfrak{S}_4$. Then

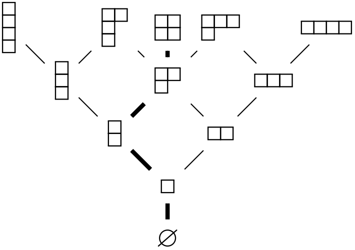
$P(w) =$

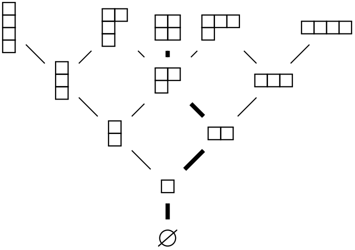
1	3
2	4

and $Q(w) =$

1	2
3	4

.





Dual graded graphs and Robinson-Schensted

- Standard Young tableaux are represented as paths from $\varnothing$ in Young's lattice.
- Through the language of *growth diagrams*, any explicit pairing of up-down and down-up paths that shows $DU - UD = I$ in any dual graded graph determines a Robinson-Schensted-like insertion algorithm.

Dual graded graphs and Robinson-Schensted

- Standard Young tableaux are represented as paths from $\varnothing$ in Young's lattice.
- Through the language of *growth diagrams*, any explicit pairing of up-down and down-up paths that shows $DU - UD = I$ in any dual graded graph determines a Robinson-Schensted-like insertion algorithm.

Question: What is the K-theoretic analogue?

Increasing and set-valued tableaux (T-Y 2009, Buch 2002)

Increasing tableaux: Rows and columns are *strictly increasing*

1	2	4	5
2	3	5	7
6	7		
8			

✓

1	2	2	4
3	4		
5			

X

Increasing and set-valued tableaux (T-Y 2009, Buch 2002)

Increasing tableaux: Rows and columns are *strictly increasing*

1	2	4	5
2	3	5	7
6	7		
8			

✓

1	2	2	4
3	4		
5			

X

Standard set-valued tableaux: Boxes filled with finite, non-empty subsets of positive integers so that rows and colums are strictly increasing and numbers $\{1, 2, \ldots, n\}$ appear exactly once.

1	2, 4	7
3, 5	6, 8	
9		

✓

1	2, 5
3	4, 6

X

$$A < B \text{ if } max(A) < min(B)$$

Hecke insertion (BKSTY 2008)

Robinson-Schensted: Permutation in $\mathfrak{S}_n \longleftrightarrow$ (SYT,SYT) of the same shape with n boxes

Hecke insertion (BKSTY 2008)

Robinson-Schensted: Permutation in $\mathfrak{S}_n \longleftrightarrow$ (SYT,SYT) of the same shape with n boxes

Hecke insertion: Word in $[n] \longleftrightarrow$ (Increasing, Standard Set-Valued) of the same shape where the number of boxes is **at most** the length of the word

Hecke insertion (BKSTY 2008)

Robinson-Schensted: Permutation in $\mathfrak{S}_n \longleftrightarrow$ (SYT,SYT) of the same shape with n boxes

Hecke insertion: Word in $[n] \longleftrightarrow$ (Increasing, Standard Set-Valued) of the same shape where the number of boxes is **at most** the length of the word

Example

Take $h = 11343242$. Then

$P_H(w) =$

1	2	4
3	4	
4		

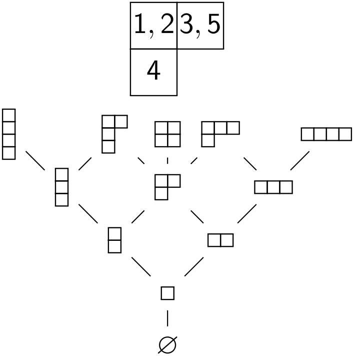
and

$Q_H(w) =$

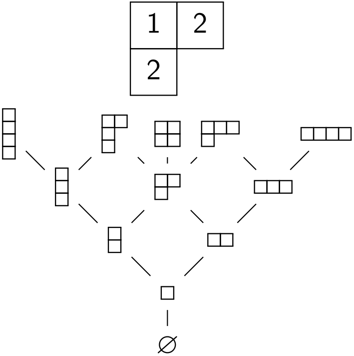
1, 2	3	4, 7
5	8	
6		

.

What is the K version of Young's lattice?

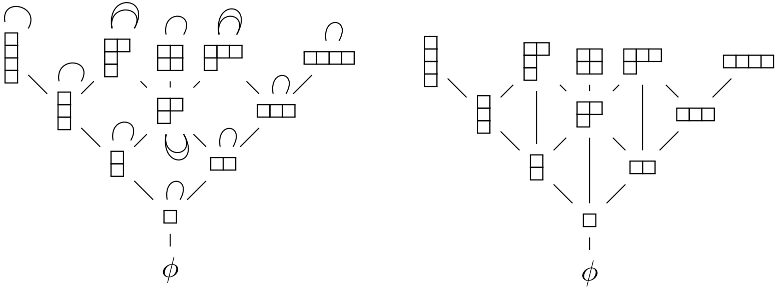


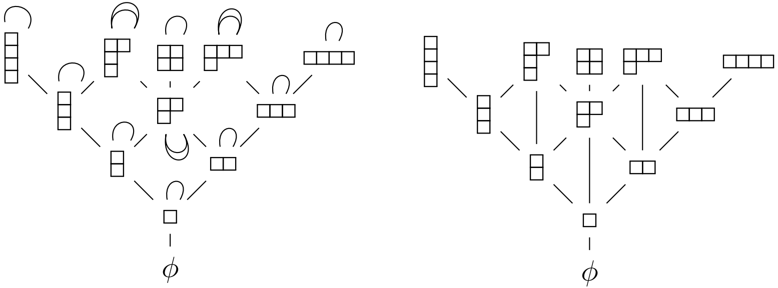
Set-valued tableaux



Increasing tableaux

What is the K version of Young's lattice?

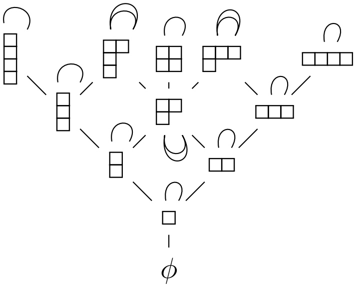


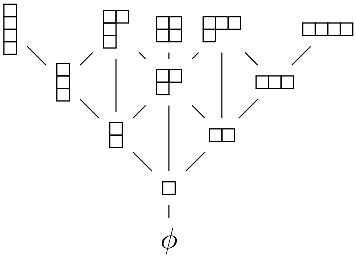


- Does this satisfy $DU - UD = I$?

$$
\begin{aligned}
(DU - UD)(\ydiagram{2}) &= D(\ydiagram{2} + \ydiagram{3} + \ydiagram{2,1}) - U(\ydiagram{1}) \\
&= (\ydiagram{1} + \ydiagram{2} + \ydiagram{1,1} + \ydiagram{1} + \ydiagram{2}) - (\ydiagram{1} + \ydiagram{1,1} + \ydiagram{2}) \\
&= \ydiagram{2} + \ydiagram{1}
\end{aligned}
$$

What is the K version of Young's lattice?





- Does this satisfy $DU - UD = I$?

$$
\begin{aligned}
(DU - UD)(\ydiagram{2}) &= D(\ydiagram{2} + \ydiagram{3} + \ydiagram{2,1}) - U(\ydiagram{1}) \\
&= (\ydiagram{1} + \ydiagram{2} + \ydiagram{1,1} + \ydiagram{1} + \ydiagram{2}) - (\ydiagram{1} + \ydiagram{1,1} + \ydiagram{2}) \\
&= \ydiagram{2} + \ydiagram{1}
\end{aligned}
$$

- Satisfies $DU - UD = D + I$.

Definition

A **weak (strict) filtered graph** is a triple $G = (P, \rho, E)$, where P is a countable set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geq \rho(x)$ (resp. $\rho(y) > \rho(x)$).

Dual Filtered Graphs

Definition

A **weak (strict) filtered graph** is a triple $G = (P, \rho, E)$, where P is a countable set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geq \rho(x)$ (resp. $\rho(y) > \rho(x)$).

- Take weak filtered graph $G_1 = (P, \rho, E_1)$ and strict filtered graph $G_2 = (P, \rho, E_2)$.

Dual Filtered Graphs

Definition

A **weak (strict) filtered graph** is a triple $G = (P, \rho, E)$, where P is a countable set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geq \rho(x)$ (resp. $\rho(y) > \rho(x)$).

- Take weak filtered graph $G_1 = (P, \rho, E_1)$ and strict filtered graph $G_2 = (P, \rho, E_2)$.
- Form an **oriented filtered graph** $G = (P, E_1, E_2)$ by orienting E_1 in the positive filtration direction and the edges of G_2 in the negative filtration direction.

Dual Filtered Graphs

Definition

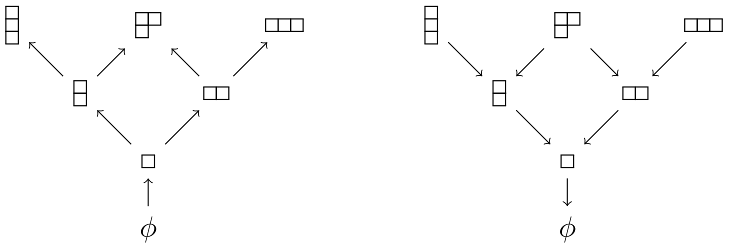
A **weak (strict) filtered graph** is a triple $G = (P, \rho, E)$, where P is a countable set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geq \rho(x)$ (resp. $\rho(y) > \rho(x)$).

- Take weak filtered graph $G_1 = (P, \rho, E_1)$ and strict filtered graph $G_2 = (P, \rho, E_2)$.
- Form an **oriented filtered graph** $G = (P, E_1, E_2)$ by orienting E_1 in the positive filtration direction and the edges of G_2 in the negative filtration direction.
- We say that G_1 and G_2 are **dual filtered graphs** if, on G,

$$DU - UD = D + I.$$

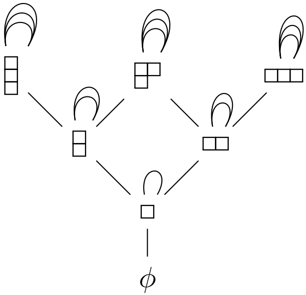
Trivial Construction

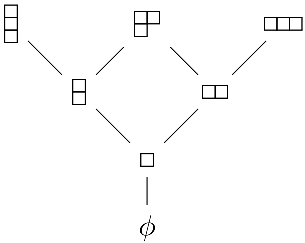
To construct a dual filtered graph from any dual graded graph, add $\rho(x)$ "upward" loops to each element x of the dual graded graph, where ρ is the rank function.



Trivial Construction

To construct a dual filtered graph from any dual graded graph, add $\rho(x)$ "upward" loops to each element x of the dual graded graph, where ρ is the rank function.





It is easy to see that this always works.

More interesting constructions

- Pieri construction - algebraic
- Möbius construction - uses Möbius function

Pieri Construction

- Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.

Pieri Construction

- Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.
  - The λ's belong to some infinite indexing set P, and each $P_i = \{\lambda | |\lambda| = i\}$ is finite.

Pieri Construction

- Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.
  - The λ's belong to some infinite indexing set P, and each $P_i = \{\lambda || \lambda| = i\}$ is finite.
  - Being filtered means that $a_\lambda a_\mu = \sum c_\nu a_\nu$, where $|\lambda| + |\mu| \leq |\nu|$.

Pieri Construction

- Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.
  - The λ's belong to some infinite indexing set P, and each $P_i = \{\lambda | |\lambda| = i\}$ is finite.
  - Being filtered means that $a_\lambda a_\mu = \sum c_\nu a_\nu$, where $|\lambda| + |\mu| \leq |\nu|$.
- Recall $D \in End(A)$ is a derivation if $D(ab) = D(a)b + aD(b)$.

Pieri Construction

Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.

Let $f = f_1 + f_2 + \ldots \in \hat{A}$ be an element of the completion of A such that $f_i \in A_i$.

Let $D \in End(A)$ be a derivation on A satisfying $D(f) = f + 1$.

Pieri Construction

Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.

- We'll take $A = \Lambda$ to be the ring of symmetric functions with the Schur function basis $\{s_\lambda\}$.

Let $f = f_1 + f_2 + \ldots \in \hat{A}$ be an element of the completion of A such that $f_i \in A_i$.

Let $D \in End(A)$ be a derivation on A satisfying $D(f) = f + 1$.

Pieri Construction

Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.

- We'll take $A = \Lambda$ to be the ring of symmetric functions with the Schur function basis $\{s_\lambda\}$.

Let $f = f_1 + f_2 + \ldots \in \hat{A}$ be an element of the completion of A such that $f_i \in A_i$.

- We'll use $f = h_1 + h_2 + h_3 + \ldots$.

Let $D \in End(A)$ be a derivation on A satisfying $D(f) = f + 1$.

Pieri Construction

Let $A = \oplus_{i \geq 0} A_i$ be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $\{a_\lambda\}$, where $a_\lambda \in A_{|\lambda|}$.

- We'll take $A = \Lambda$ to be the ring of symmetric functions with the Schur function basis $\{s_\lambda\}$.

Let $f = f_1 + f_2 + \ldots \in \hat{A}$ be an element of the completion of A such that $f_i \in A_i$.

- We'll use $f = h_1 + h_2 + h_3 + \ldots$.

Let $D \in End(A)$ be a derivation on A satisfying $D(f) = f + 1$.

- For us, $D = \frac{d}{dp_1}$.

Pieri Construction

To form a graph from fixed A, f, and D:

Upward edges from λ to $\mu \longleftrightarrow$ (coefficient of a_λ in $D(a_\mu)$)

Pieri Construction

To form a graph from fixed A, f, and D:

Upward edges from λ to $\mu \longleftrightarrow$ (coefficient of a_λ in $D(a_\mu)$)

Downward edges from μ to $\lambda \longleftrightarrow$ (coefficient of a_μ in $a_\lambda f$)

Pieri Construction

To form a graph from fixed A, f, and D:

Upward edges from λ to $\mu \longleftrightarrow$ (coefficient of a_λ in $D(a_\mu)$)

Downward edges from μ to $\lambda \longleftrightarrow$ (coefficient of a_μ in $a_\lambda f$)

- Upward edges correspond $\longleftrightarrow$ coefficient of s_λ in $D(s_\mu) = \frac{d}{dp_1} s_\mu$
- downward edges $\longleftrightarrow$ coefficient of s_μ in $s_\lambda(h_1 + h_2 + h_3 + \ldots)$.

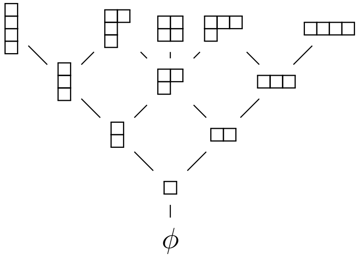
Pieri Construction

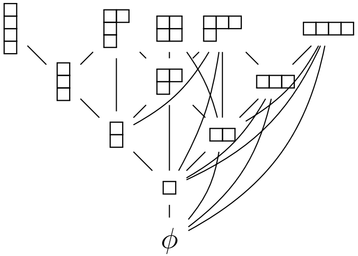
To form a graph from fixed A, f, and D:

Upward edges from λ to $\mu \longleftrightarrow$ (coefficient of a_λ in $D(a_\mu)$)

Downward edges from μ to $\lambda \longleftrightarrow$ (coefficient of a_μ in $a_\lambda f$)

- Upward edges correspond $\longleftrightarrow$ coefficient of s_λ in $D(s_\mu) = \frac{d}{dp_1} s_\mu$
- Downward edges $\longleftrightarrow$ coefficient of s_μ in $s_\lambda(h_1 + h_2 + h_3 + \ldots)$.





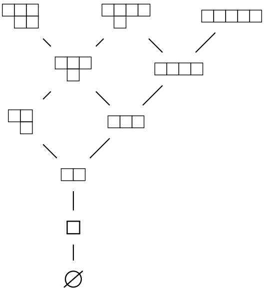
Pieri Construction

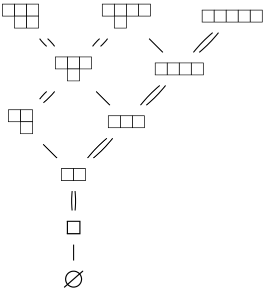
Theorem (P.-Pylyavskyy)

The Pieri construction yields a dual filtered graph.

Remark: If we begin with a bialgebra A, there is a natural choice for derivation D.

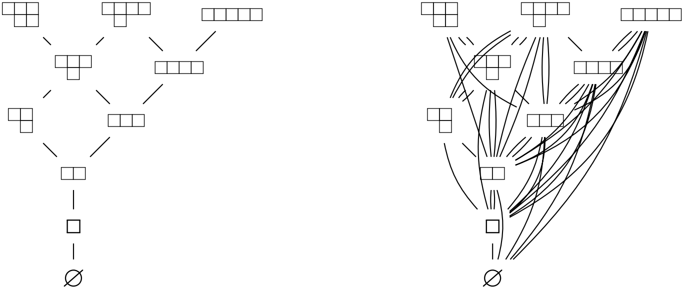
Another Pieri construction: shifted Young's lattice





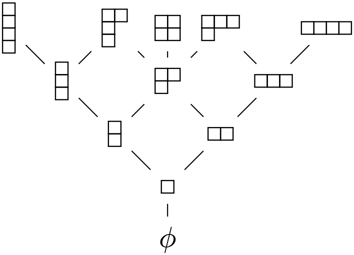
$A = \Lambda'$ is the subring of the ring of symmetric functions generated by $\{p_1, p_3, p_5, \ldots\}$.

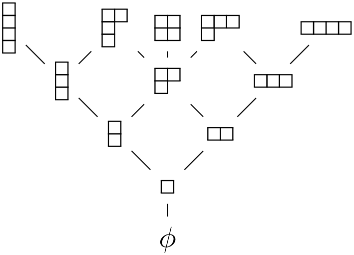
Another Pieri construction: shifted Young's lattice



$A = \Lambda'$ is the subring of the ring of symmetric functions generated by $\{p_1, p_3, p_5, \ldots\}$.

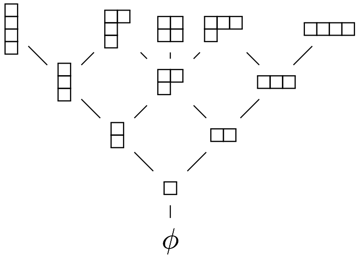
Start with a dual graded graph G, and form a pair of filtered graphs as follows.





Möbius Construction

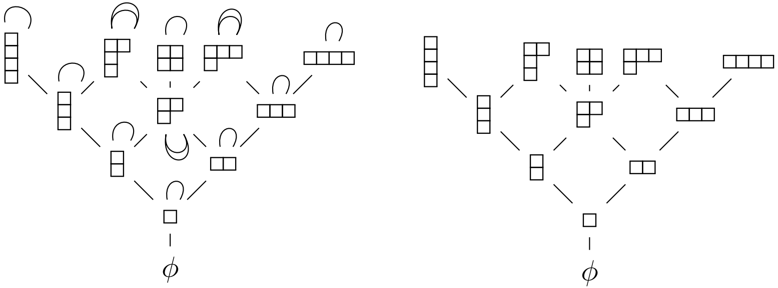
Start with a dual graded graph G, and form a pair of filtered graphs as follows.



- Add an "upward oriented" loop to each vertex y for each upward oriented edge ending at y.

Möbius Construction

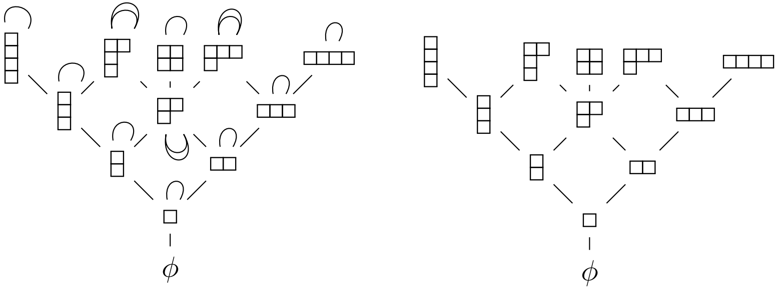
Start with a dual graded graph G, and form a pair of filtered graphs as follows.



- Add an "upward oriented" loop to each vertex y for each upward oriented edge ending at y.

Möbius Construction

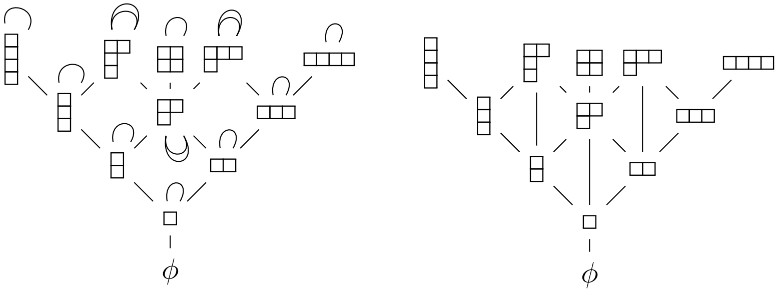
Start with a dual graded graph G, and form a pair of filtered graphs as follows.



- Add an "upward oriented" loop to each vertex y for each upward oriented edge ending at y.
- If $(x, y) \notin E_2$, add $|\mu(x, y)|$ edges between vertices x and y, where μ is the Möbius function in (P, ρ, E_2).

Möbius Construction

Start with a dual graded graph G, and form a pair of filtered graphs as follows.

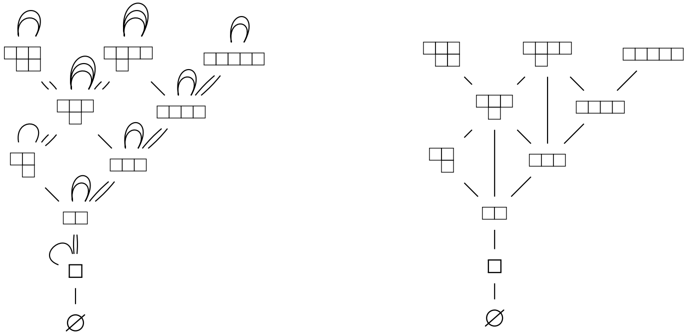


- Add an "upward oriented" loop to each vertex y for each upward oriented edge ending at y.
- If $(x, y) \notin E_2$, add $|\mu(x, y)|$ edges between vertices x and y, where μ is the Möbius function in (P, ρ, E_2).

Möbius Construction

- The Möbius construction does not always produce a pair of dual filtered graphs, and it is unclear how to determine when it does.
- In some major examples, however, it is the result of this construction that relates to natural K-theoretic analogues of Robinson-Schensted-like algorithms.

Möbius Construction example: shifted Young's lattice

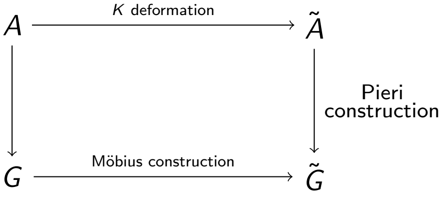


Using this dual filtered graph, we constructed a shifted Hecke insertion, a natural K-theoretic analogue of the shifted insertion of Sagan and Worley. We conjecture this shifted Hecke insertion coincides with the shifted K-jdt (Clifford-Thomas-Yong).

Möbius via Pieri phenomenon

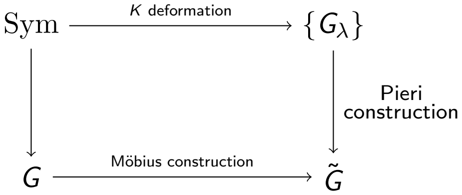
Möbius via Pieri phenomenon

- Let A be a graded Hopf algebra, and let bialgebra $\tilde{A}$ be its *K-theoretic deformation* in some appropriate (but mysterious) sense.
- Let G be a natural dual graded graph associated with A.
- Applying the Möbius construction to G yields *the same result* as a natural Pieri construction applied to $\tilde{A}$.



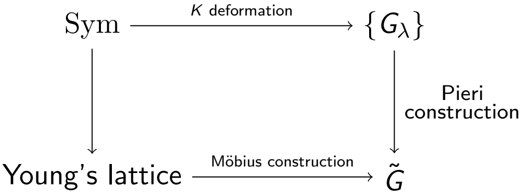
Möbius via Pieri phenomenon

- Let A be a graded Hopf algebra, and let bialgebra $\tilde{A}$ be its *K-theoretic deformation* in some appropriate (but mysterious) sense.
- Let G be a natural dual graded graph associated with A.
- Applying the Möbius construction to G yields *the same result* as a natural Pieri construction applied to $\tilde{A}$.



Möbius via Pieri phenomenon

- Let A be a graded Hopf algebra, and let bialgebra $\tilde{A}$ be its *K-theoretic deformation* in some appropriate (but mysterious) sense.
- Let G be a natural dual graded graph associated with A.
- Applying the Möbius construction to G yields *the same result* as a natural Pieri construction applied to $\tilde{A}$.



Future work

- What other polynomials in U and D can appear on the right-hand side? (Formal group laws)
- When does the Möbius construction work?
- How can we modify the Möbius construction for graphs with multiple edges?
- Why does the Möbius via Pieri phenomenon happen?
- Can we rephrase the Pieri construction in Hopf algebra language? (Group-like elements and characters)

References

- A. Buch, A Littlewood Richardson rule for the K-theory of Grassmannians (2002)
- A. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong, Stable Grothendieck polynomials and K-theoretic factor sequences (2008)
- A. Buch and M. Samuel, K-theory of miniscule varieties (2013)
- E. Clifford, H. Thomas, and A. Yong, K-theoretic Schubert calculus for $\mathrm{OG}(n, 2n+1)$ and jeu de taquin for shifted increasing tableaux (2010)
- S. Fomin, Duality of Graded Graphs (1994)
- T. Lam and P. Pylyavskyy, Combinatorial Hopf Algebras and K-Homology of Grassmannians (2007)
- R. Patrias and P. Pylyavskyy, Dual filtered graphs, arXiv 1410.7683
- R. Patrias and P. Pylyavskyy, K-theoretic Poirier-Reutenauer bialgebra, arXiv 1404.4340
- H. Thomas and A. Yong, A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus (2009)