Sign variation, the Grassmannian, and total positivity

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FPSAC 2015
KAIST, Daejeon
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Given $V \in \text{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $I \in \binom{[n]}{k}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ with columns $I$.

The Plücker coordinates $\Delta_I(V)$ are well-defined up to multiplication by a global nonzero constant.

We say that $V \in \text{Gr}_{k,n}$ is totally nonnegative if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$.

Denote the set of such $V$ by $\text{Gr}_{\geq 0}^{k,n}$, called the totally nonnegative Grassmannian.

$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & -2 \end{bmatrix} \in \text{Gr}_{2,4}$

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![Diagram of Grassmannian](image)

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Let $V \in \text{Gr}_{k,n}$. Then $V$ is totally nonnegative iff $\text{var}(x) \leq k - 1$ for all $x \in V$. 

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- Note that every $V \in \text{Gr}_{k,n}$ contains a vector $x$ with $\text{var}(x) \geq k - 1$. So, the totally nonnegative subspaces are those whose vectors change sign as few times as possible.
A history of total positivity

- Descartes's rule of signs (1637): The number of positive real zeros of a real polynomial is at most the number of sign changes of its sequence of coefficients.
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- Postnikov (2006) studied $\text{Gr}_{k,n}^{\geq 0}$ from a combinatorial perspective.
How close is a subspace to being totally nonnegative?

Can we determine $\max_{x \in V} \text{var}(x)$ from the Plücker coordinates of $V$?

**Theorem (Karp (2015))**

Let $V \in \text{Gr}_{k,n}$ and $m \geq k - 1$.

(i) If $\text{var}(x) \leq m$ for all $x \in V$, then $\text{var}(\Delta J \cup \{i\} (V))_{i \in J} \leq m - k + 1$ for all $J \in \binom{[n]}{k-1}$.

The converse holds if $V$ is generic (i.e. $\Delta I (V) \neq 0$ for all $I$).

(ii) We can perturb $V$ into a generic $W$ with $\max_{x \in V} \text{var}(x) = \max_{x \in W} \text{var}(x)$. 

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**Proof:**

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(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (2, 1, 1), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (1, 4, -6),
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The 4 sequences of Plücker coordinates are:

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How close is a subspace to being totally nonnegative?

- Can we determine \( \max_{x \in V} \text{var}(x) \) from the Plücker coordinates of \( V \)?

**Theorem (Karp (2015))**

Let \( V \in \text{Gr}_{k,n} \) and \( m \geq k - 1 \).

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The converse holds if \( V \) is generic (i.e. \( \Delta_I(V) \neq 0 \) for all \( I \)).

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The totally positive Grassmannian

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- For $x \in \mathbb{R}^n$, let $\overline{\text{var}}(x)$ be the maximum of $\text{var}(y)$ over all $y \in \mathbb{R}^n$ obtained from $x$ by changing zero components of $x$. 
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Oriented matroids

An **oriented matroid** is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.

These results generalize to oriented matroids.
The cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$

Given $V \in \text{Gr}_{k,n}$, define $M(V) := \{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \}$, called the matroid of $V$. The matroid stratification of $\text{Gr}_{k,n}^{\geq 0}$ is a CW-decomposition.
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$\text{Gr}_{1,3}^{\geq 0} \cong \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$
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How can we find the cell of $V$ (i.e. $M(V)$) in $\text{Gr}_{k,n}^{\geq 0}$ using sign patterns?
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**Exercise**

Let $V \in \text{Gr}_{k,n}$ and $I \in \binom{[n]}{k}$. Then $\Delta_I(V) \neq 0$ iff $V$ realizes all $2^k$ sign patterns in $\{+,-\}^k$ on $I$. 

Steven N. Karp (UC Berkeley)

Sign variation, the Grassmannian, and total positivity

FPSAC 2015
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- Moreover, given $\omega \in \{+, -\}^k$, there exists $V \in \text{Gr}_{k,n}$ which realizes all $2^k$ sign patterns in $\{+, -\}^k$ on $I$ except for $\pm \omega$ (assuming $n > k$).
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Let $V \in \operatorname{Gr}_{k,n}^{\geq 0}$ and $I \in \binom{[n]}{k}$. Then $\Delta_I(V) \neq 0$ iff $V$ realizes the following $k$ sign patterns on $I$:

$(+, -, +, -, +, -, \cdots), (+, +, -, +, -, +, \cdots), (+, -, -, +, -, +, \cdots), \cdots$.

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- Given $V \in \text{Gr}_{k,n}$, define $M(V) := \{ I \in \binom{[n]}{k} : \Delta_I(V) \neq 0 \}$, called the matroid of $V$. The matroid stratification of $\text{Gr} \geq 0_{k,n}$ is a CW-decomposition.

\[ \text{Gr} \geq 0_{1,3} \cong \{1\}, \{2\}, \{1\}, \{2\}, \{1\}, \{3\} \]

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\[ (+, -, +, -, +, - ,\ldots), (+, +, -, +, - ,\ldots), (+, -, -, +, - ,\ldots), \ldots. \]

- Compare this to the fact that the matroid stratification of $\text{Gr} \geq 0_{k,n}$ is the refinement of $n$ cyclically shifted Schubert stratifications (vs. all $n!$).
Further directions

- Is there an efficient way to test whether a given $V \in \text{Gr}_k,n$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether $V$ is totally positive, we only need to check that some particular $k(n - k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)
- Is there a simple way to index the cell decomposition of $\text{Gr}_{k,n}^{\geq 0}$ using the data of sign patterns?
- Is there a nice stratification of the subset of the Grassmannian
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Thank you!