Sign variation, the Grassmannian, and total positivity

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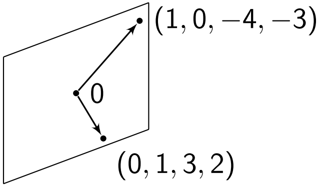
$V :=$ 

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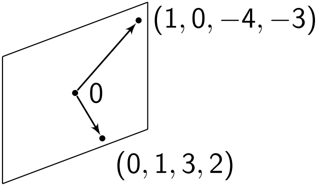
$V :=$ 

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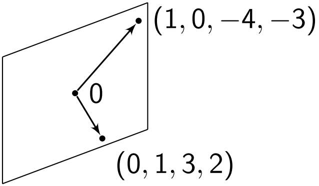
$V :=$ 

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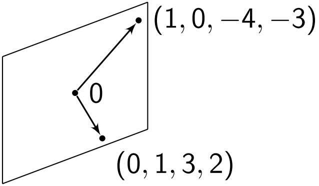
$V :=$ 

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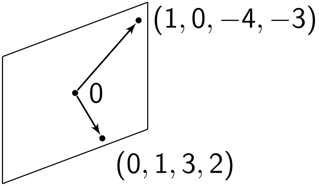

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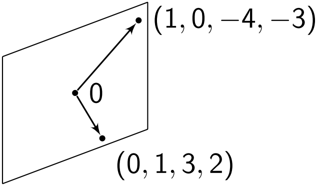

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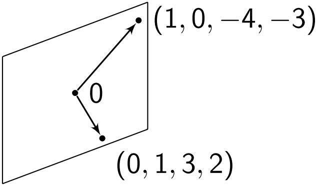

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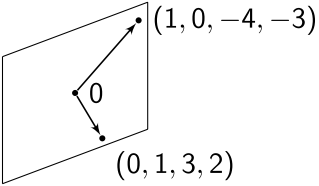
$V :=$ 

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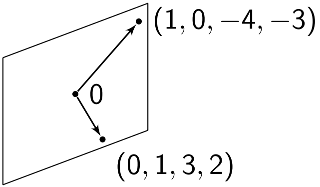
- e.g. $V :=$ 

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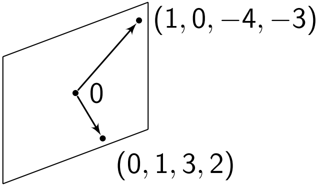
- e.g. $V :=$ 

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- Note that every $V \in \mathrm{Gr}_{k,n}$ contains a vector x with $\mathrm{var}(x) \ge k - 1$. So, the totally nonnegative subspaces are those whose vectors change sign as few times as possible.

A history of total positivity

- Descartes's rule of signs (1637): The number of positive real zeros of a real polynomial is at most the number of sign changes of its sequence of coefficients.

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- Postnikov (2006) studied $\mathrm{Gr}_{k,n}^{\ge 0}$ from a combinatorial perspective.

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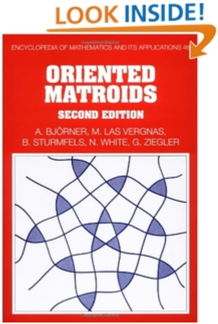
- Note that var is *increasing* while $\overline{\mathrm{var}}$ is *decreasing* with respect to genericity.

Oriented matroids

- An *oriented matroid* is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.



- These results generalize to oriented matroids.

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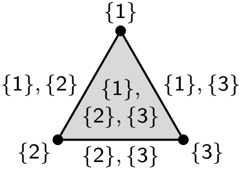

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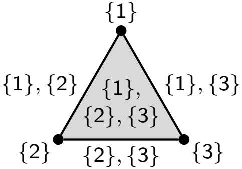

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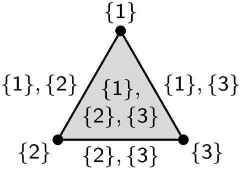
$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ 

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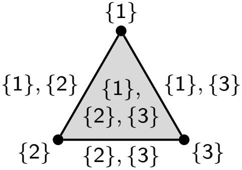

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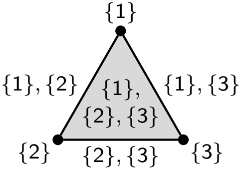

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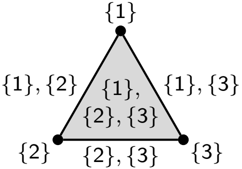
$\mathrm{Gr}_{1,3}^{\geq 0} \cong$ 

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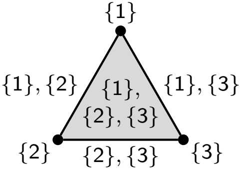

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- Compare this to the fact that the matroid stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$ is the refinement of n cyclically shifted *Schubert stratifications* (vs. all $n!$).

Further directions

- Is there an efficient way to test whether a given $V \in \mathrm{Gr}_{k,n}$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether V is totally positive, we only need to check that some particular $k(n-k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)
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