Jack symmetric functions and graphs on surfaces

Valentin Féray

Institut für Mathematik, Universität Zürich

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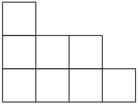
Outline of the talk

1. Background on (Jack) symmetric functions
2. Hanlon's conjecture
3. Goulden-Jackson's conjecture
4. Lassalle's conjecture
5. Link between the conjectures
6. Proof of one result

Symmetric functions

- partitions: $(4,3,1) \leftrightarrow$  .
- monomial symmetric functions

$$m_{(2,1)}(x_1, x_2, \dots) = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + \dots$$

- power-sum symmetric functions

$$p_{(2,1)}(x_1, x_2, \dots) = (x_1^2 + x_2^2 + \dots)(x_1 + x_2 + \dots).$$

- Schur symmetric functions

$$s_{(2,1)}(x_1, x_2, \dots) = m_{(2,1)}(x_1, x_2, \dots) + 2m_{(1^3)}(x_1, x_2, \dots).$$

(defined as sum over tableaux, quotient of determinants or from representation theory)

A characterization of Schur symmetric functions

Hall scalar product is defined by $\langle p_\mu, p_\nu \rangle := z_\mu \delta_{\mu,\nu}$.

Proposition

The basis (s_λ) is the unique family of symmetric functions with the following properties:

1. triangularity: $s_\lambda = \sum_{\nu \preceq_d \lambda} c_\nu^\lambda m_\nu$;
2. orthogonality: $\langle s_\lambda, s_\mu \rangle = 0$ if $\lambda \neq \mu$;
3. normalization: $\langle s_\lambda, s_\lambda \rangle = 1$ and $[m_{(1^n)}] s_\lambda > 0$.

z_μ: standard numerical factor;

dominance order: $\nu \preceq_d \lambda$ if and only if $|\nu| = |\lambda|$ and

$$\text{for all } i, \quad \nu_1 + \cdots + \nu_i \leq \lambda_1 + \cdots + \lambda_i.$$

Jack polynomials

Consider the following deformation of Hall scalar product:

$$\langle p_\mu, p_\nu \rangle_\alpha = \alpha^{\ell(\mu)} z_\mu \delta_{\mu,\nu}$$

$\ell(\mu)$: length (number of parts) of the partition μ.

Jack polynomials

Consider the following deformation of Hall scalar product:

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Definition

Jack polynomials $J_\lambda^{(\alpha)}$ is the unique family of symmetric functions with the following properties:

1. triangularity: $J_\lambda^{(\alpha)} = \sum_{\nu \preceq_d \lambda} c_\nu^\lambda m_\nu$;
2. orthogonality: $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle = 0$ if $\lambda \neq \mu$;
3. normalization: $\left[m_{(1^{|\lambda|})}\right] J_\lambda^{(\alpha)} = 1$.

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3. normalization: $\left[m_{(1^{|\lambda|})}\right] J_\lambda^{(\alpha)} = 1$.

Specialization: $J_\lambda^{(1)} = H_\lambda s_\lambda$.
H_λ: combinatorial factor (product of hooks of λ).

Transition

Hanlon's conjecture

A formula for Schur functions

Choose a filling T_0 of a Young diagram λ.

Example:

$$\lambda = (2,2), \quad T_0 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}.$$

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Choose a filling T_0 of a Young diagram λ. Define

$$\mathrm{RS}(T_0) = \text{row stabilizer of } T_0\ ;$$

$$\mathrm{CS}(T_0) = \text{column stabilizer of } T_0.$$

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$$\mathrm{RS}(T_0) = \{\mathrm{id}, (1\ 3), (2\ 4), (1\ 3)(2\ 4)\}$$
$$\mathrm{CS}(T_0) = \{\mathrm{id}, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$$

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Proposition (folklore? Hanlon, 1988?)

$$H_\lambda s_\lambda = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\, p_{\mathrm{type}(\sigma\tau)}$$

(type = cycle-type)

Hanlon's conjecture

Conjecture (Hanlon, 1988)

There exists a weight function $(\sigma, \tau) \mapsto w(\sigma, \tau)$ (that fulfills some technical conditions) such that

$$J_\lambda^{(\alpha)} = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \alpha^{w(\sigma,\tau)} \varepsilon(\tau)\, p_{\mathrm{type}(\sigma\tau)}$$

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The case $\alpha = 2$

Definition

A pair-partition of $2n$ is a partition of $[2n] = \{1, \dots, 2n\}$ into 2-element sets.

Let λ be a Young diagram and T_0 a fixed filling of 2λ.
Denote:

- $\mathrm{RS}^{(2)}(T_0) =$ set of pair-partitions that match elements in the same row.
- $\mathrm{CS}^{(2)}(T_0) =$ set of pair-partitions that match elements in column $2i+1$ with elements in column $2i+2$.

Let $\lambda = (2,1)$ and $T_0 = \begin{array}{|c|c|cc} \hline 5 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$. Then

$$\mathrm{RS}^{(2)}(T_0) = \{12|34|56,\ 13|24|56,\ 14|23|56\}$$
$$\mathrm{CS}^{(2)}(T_0) = \{12|34|56,\ 16|34|25\}$$

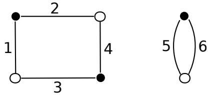
The case $\alpha = 2$

Theorem (F., Śniady, 2011)

$$J_\lambda^{(2)} = \sum_{\substack{S_1 \in \mathrm{RS}^{(2)}(T_0) \\ S_2 \in \mathrm{CS}^{(2)}(T_0)}} \varepsilon^{(2)}(S_2)\, p_{\mathrm{type}(S_1,S_2)}$$

Type of a pair of pair-partitions:

$S_1 = 12|34|56;$
$S_2 = 13|24|56.$



$\mathrm{type}(S_1, S_2) = (2, 1).$

$\varepsilon^{(2)}(T)$: analog of the sign of permutations.

Transition

- For $\alpha = 2$, rather than adding a weight, it is more natural to work with different combinatorial objects: pair-partitions instead of permutations.
- We will see that both formulas (for $\alpha = 1$ and $\alpha = 2$) have interpretations in terms of graphs embedded in surfaces.

Pair of permutations and graphs embedded in surfaces

Classical bijection between

$$S_k \times S_k \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \text{union of bicolored} \\ \text{labeled oriented maps} \end{array} \right\}.$$

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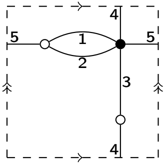

map = connected graph embedded in a surface (up to isomorphism with a technical condition).
oriented (map) = in an oriented surfaces.
labeled = edges labeled from 1 to k.

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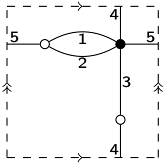
$$\begin{array}{c} \sigma = (1\ 5\ 2)(3\ 4) \\ \tau = (1\ 2\ 3\ 5\ 4) \end{array} \quad \longleftrightarrow$$



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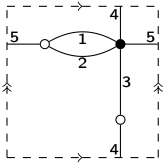
$$\begin{array}{r} \sigma = (1\ 5\ 2)(3\ 4) \\ \tau = (1\ 2\ 3\ 5\ 4) \\ \sigma\tau = (1)(2\ 4\ 5\ 3) \end{array} \longleftrightarrow$$



- cycle-type of $\sigma \leftrightarrow$ white vertex degree distribution of the map(s);
- cycle-type of $\tau \leftrightarrow$ black vertex degree distribution of the map(s);
- cycle-type of the product $\sigma\tau \leftrightarrow$ face degree distribution of the map(s).

Proposition (folklore? Hanlon, 1988?) reformulated

$$H_\lambda s_\lambda = \sum_M (-1)^{k-|V_\bullet(M)|} p_{\text{face-type}(M)},$$

where the sum runs over union of oriented labeled bicolored maps M with $V_\circ(M) \leq \text{Rows}(T)$ and $V_\bullet(M) \leq \text{Cols}(T)$.

$\leq$: refinement of set-partitions. (face-type = face degree distribution).

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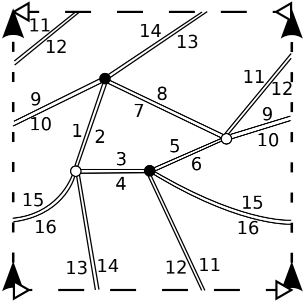
The case $\alpha = 2$ admits a similar reformulation, since there is a bijection between union of bicolored labeled non-oriented maps and triples of pair-partitions.

non-oriented maps = connected graph embedded in non-oriented surface.

Maps on non-oriented surfaces and triple of pair-partitions



A map on the Klein bottle.

$$S_0 = 1,2|3,4|5,6|7,8|9,10|11,12|13,14|15,16;$$
$$S_1 = 1,15|2,3|4,14|13,16|5,7|6,10|8,11|9,12;$$
$$S_2 = 1,10|2,7|8,13|9,14|3,5|4,12|6,15|11,16.$$

Transition

Goulden-Jackson's conjecture

Frobenius counting formula

Theorem (Frobenius counting formula)

Let μ, ν and ρ be partitions of n. Let $C^{\rho}_{\mu,\nu}$ the number of pairs (σ, τ) such that

- σ and τ have cycle-type μ and ν, respectively;
- $\sigma\,\tau$ has cycle-type ρ.

Then

$$|C^{\rho}_{\mu,\nu}| = \frac{n!}{z_\mu\, z_\nu\, z_\rho} \sum_{\lambda \vdash n} H_\lambda\, \chi^\lambda_\mu\, \chi^\lambda_\nu\, \chi^\lambda_\rho.$$

χ^λ_μ: irreducible character value of symmetric groups.

This is a classical result of representation theory.

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Recall that $s_\lambda = \sum_\mu \chi^\lambda_\mu \frac{p_\mu}{z_\mu}$. Consider three disjoint infinite alphabets $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. Then Frobenius formula can be written as

$$\sum_{\lambda \vdash n} H_\lambda\, s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) s_\lambda(\mathbf{z}) = \sum_{\mu,\nu,\rho \vdash n} \frac{|C^{\rho}_{\mu,\nu}|}{n!} p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\rho(\mathbf{z}).$$

Frobenius counting formula and oriented maps

$$\sum_{\lambda \vdash n} H_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) s_\lambda(\mathbf{z}) = \sum_{\mu,\nu,\rho \vdash n} \frac{|C^{\rho}_{\mu,\nu}|}{n!} p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\rho(\mathbf{z}).$$

But $|C^{\rho}_{\mu,\nu}|$ counts union of bicolored oriented maps with (vertex/face) degree distributions μ, ν and ρ.

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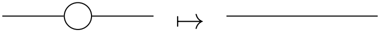
$$\sum_{\lambda \vdash n} H_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) s_\lambda(\mathbf{z}) = \sum_{\mu,\nu,\rho \vdash n} \frac{|C_{\mu,\nu}^{\rho}|}{n!} p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\rho(\mathbf{z}).$$

But $|C_{\mu,\nu}^{\rho}|$ counts union of bicolored oriented maps with (vertex/face) degree distributions μ, ν and ρ.

If n is odd and $\nu = (2^{n/2})$, we count bicolored maps with white vertices of degree 2. The latter are in easy bijection with (monocolored) maps



Frobenius counting formula and oriented maps

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But $|C_{\mu,\nu}^{\rho}|$ counts union of bicolored oriented maps with (vertex/face) degree distributions μ, ν and ρ.

We would prefer to count connected objects rather than unions!

$$\log\left(\sum_{n \geq 0} t^n \sum_{\lambda \vdash n} H_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) s_\lambda(\mathbf{z})\right) = \sum_{n \geq 1} \frac{t^n}{n!} \left(\sum_{\mu,\nu,\rho} b_{\mu,\nu,\rho}\, p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\rho(\mathbf{z})\right).$$

where $b_{\mu,\nu,\rho}$ counts bicolored oriented maps with (vertex/face) degree distributions μ, ν and ρ.

The case $\alpha = 2$

Theorem (Goulden, Jackson, 1996)

$$\log\left(\sum_{\substack{n\geq 0\\ \lambda\vdash n}} t^n \frac{J_\lambda^{(2)}(\mathbf{x})J_\lambda^{(2)}(\mathbf{y})J_\lambda^{(2)}(\mathbf{z})}{\left\langle J_\lambda^{(2)}, J_\lambda^{(2)}\right\rangle_2}\right) = \sum_{n\geq 1}\frac{t^n}{n!}\left(\sum_{\mu,\nu,\rho} b_{\mu,\nu,\rho}^{(2)}\, p_\mu(\mathbf{x})p_\nu(\mathbf{y})p_\rho(\mathbf{z})\right),$$

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where $b_{\mu,\nu,\rho}^{(2)}$ counts bicolored labeled non-oriented maps with (vertex/face) degree distributions μ, ν and ρ.

And for a generic value of α?

Define $b^{(\alpha)}_{\mu,\nu,\rho}$ by

$$\log\left(\sum_{\substack{n\geq 0\\ \lambda\vdash n}} t^n \frac{J^{(\alpha)}_\lambda(\mathbf{x}) J^{(\alpha)}_\lambda(\mathbf{y}) J^{(\alpha)}_\lambda(\mathbf{z})}{\left\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda \right\rangle_\alpha}\right) = \sum_{n\geq 1} \frac{t^n}{n!}\left(\sum_{\mu,\nu,\rho} b^{(\alpha)}_{\mu,\nu,\rho}\, p_\mu(\mathbf{x}) p_\nu(\mathbf{y}) p_\rho(\mathbf{z})\right).$$

Conjecture (Goulden-Jackson, 1996)

1. $b^{(\alpha)}_{\mu,\nu,\rho}$ is a polynomial with nonnegative coefficient in $\beta := \alpha - 1$;
2. More precisely, there exists a statistics $w(M)$ with nonnegative integer values such that
$$b^{(\alpha)}_{\mu,\nu,\rho} = \sum_M (\alpha - 1)^{w(M)},$$
where the sum runs over bicolored labeled non-oriented maps with (vertex/face) degree distributions μ, ν and ρ.

Some results on Goulden-Jackson's conjecture

1. Brown-Jackson (2007)/Lacroix (2009)/Kanunnikov-Vassilieva (2014): different special cases using different weights.

Some results on Goulden-Jackson's conjecture

1. Brown-Jackson (2007)/Lacroix (2009)/Kanunnikov-Vassilieva (2014): different special cases using different weights.
2. Dołęga-Féray (in preparation): $b^{(\alpha)}_{\mu,\nu,\rho}$ is a polynomial in α.

Transition

Lassale's conjecture

The case $\alpha = 1$

Theorem (F., Śniady 2011, Conjecture of Stanley)

For any partition μ of k without parts equal to 1,

$$\left[p_{\mu 1^{n-k}}\right](H_\lambda s_\lambda) = \frac{(-1)^k}{k!} \sum_M (-1)^{|V_\circ(M)|} N_M(\lambda),$$

where the sum runs over union of bicolored labeled oriented maps of face-type μ and $N_M(\lambda)$ is defined below.

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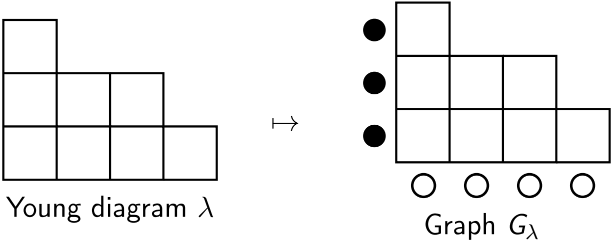

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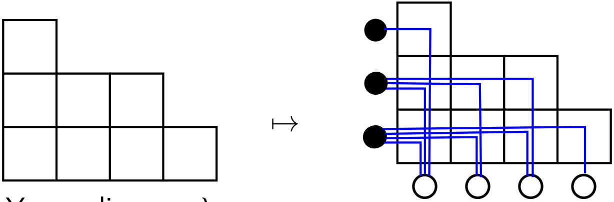

Young diagram λ $\mapsto$ Graph G_λ

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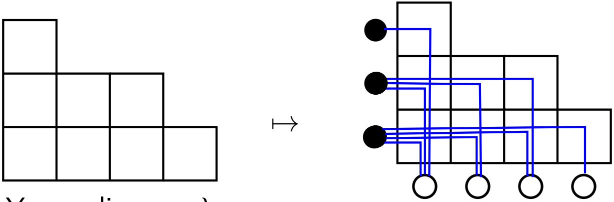

Young diagram λ $\mapsto$ Graph G_λ

Then $N_M(\lambda)$ counts bicolored graph homomorphisms from M to G_λ.

The case $\alpha = 2$

Theorem (F., Śniady, 2011)

For any partition μ of k,

$$\left[p_{\mu 1^{n-k}}\right](J_\lambda^{(2)}) = \frac{(-1)^k}{(2k)!} \sum_M (-2)^{|V_\circ(M)|} N_M(\lambda),$$

where the sum runs over union of bicolored labeled non-oriented maps of face-type μ.

Lassalle's conjecture

Conjecture (Lassalle, 2009/F., Dołęga, Śniady, 2014)

To each M, we can associate a polynomial $wt_M(\gamma)$ with nonnegative coefficients such that:

$$\frac{(-1)^{\ell(\pi)}(2k!)}{2^{\ell(\mu)}\sqrt{\alpha}^{k-\ell(\mu)}}\left[p_{\mu 1^{n-k}}\right](J_\lambda^{(\alpha)}) = \sum_M (-1)^{|V_\bullet(G)|}\, wt_M\left(\frac{1-\alpha}{\sqrt{\alpha}}\right) N_M^{(\alpha)}(\lambda),$$

where the sum runs over union of bicolored labeled non-oriented maps of face-type μ and

$$N_M^{(\alpha)}(\lambda) := \left(\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M)|} (\sqrt{\alpha})^{|V_\circ(M)|}\, N_M(\lambda).$$

Partial results

- Lassalle 2009, F., Dołęga, Śniady, 2014: a weight that works for multirectangular Young diagram λ.
- F., Dołęga, 2015: polynomiality in $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$.

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But our weight is not valid for a general partition λ from $\mu = (9)$.

Transition

Link between the conjectures

Hanlon and Lassalle's conjecture

In the case $\alpha = 1$ and $\alpha = 2$ the formulas for $\left[p_{\mu 1^{n-k}}\right](J_\lambda^{(\alpha)})$ (special cases of Lassalle's conjecture) are deduced from the formulas for $J_\lambda^{(\alpha)}$ (Hanlon's formula and its analogue).

But *a priori* no implication for general α (not even a unified way to deal with $\alpha = 1$ and $\alpha = 2$).

Goulden-Jackson's and Lassalle's conjecture

Even in the case $\alpha = 1$ and $\alpha = 2$, we do not know how to start from one result to prove the other.

Goulden-Jackson's and Lassalle's conjecture

Even in the case $\alpha = 1$ and $\alpha = 2$, we do not know how to start from one result to prove the other.

Yet, the weights solving particular cases are similar.

A recent example:

- Śniady found a formula for top-degree terms (for some unusual gradation) in Lassalle's conjecture (two weeks ago on arXiv);
- Then Dołęga proved a similar formula for top-degree terms in Goulden-Jackson conjecture (in preparation).

Transition

Proof of one result

A representation-theory free proof of Hanlon's formula for Schur functions

We want to prove:

Proposition (folklore? Hanlon, 1988?)

For any partition λ of k, there exists a constant C_λ such that

$$s_\lambda = C_\lambda \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\, p_{\mathrm{type}(\sigma\tau)}.$$

Call t_λ the sum in the right hand-side. It is enough to show:

1. triangularity: $t_\lambda = \sum_{\nu \preceq_d \lambda} c_\nu^\lambda m_\nu$.
2. orthogonality: $\langle t_\lambda, t_\mu \rangle = 0$ if $\lambda \neq \mu$.
3. $t_\lambda \neq 0$.

3 is trivial, let us show 1 and 2.

Proof of triangularity: $t_\lambda = \sum_{\nu \preceq_d \lambda} c_\nu^\lambda n_\nu$ (1/2)

By definition,

$$t_\lambda = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\, p_{\mathrm{type}(\sigma\tau)}.$$

We should write this in monomial basis

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We should write this in monomial basis

$$t_\lambda = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau) \left(\sum_{\pi \geq C(\sigma\tau)} m_{\mathrm{type}(\pi)} \right),$$

where the sum runs over set partitions π that are coarser than $\sigma\tau$.
type(π): sizes of the blocks of π in nonincreasing order.

Proof of triangularity: $t_\lambda = \sum_{\nu \preceq_d \lambda} c_\nu^\lambda n_\nu$ (1/2)

By definition,

$$t_\lambda = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau)\, p_{\mathrm{type}(\sigma\tau)}.$$

We should write this in monomial basis

$$t_\lambda = \sum_{\substack{\sigma \in \mathrm{RS}(T_0) \\ \tau \in \mathrm{CS}(T_0)}} \varepsilon(\tau) \left(\sum_{\pi \geq C(\sigma\tau)} m_{\mathrm{type}(\pi)} \right),$$

where the sum runs over set partitions π that are coarser than $\sigma\tau$.
type(π): sizes of the blocks of π in nonincreasing order.
Thus,

$$t_\lambda = \sum_\pi m_{\mathrm{type}(\pi)} \left[\sum_{\substack{\sigma \in \mathrm{RS}(T_0), \tau \in \mathrm{CS}(T_0) \\ C(\sigma\tau) \leq \pi}} \varepsilon(\tau) \right].$$

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Now $\tau \leftrightarrow \tau\,(i,j)$ is a sign-reversing involution that shows

$$\left[\sum_{\substack{\sigma \in \mathrm{RS}(T_0), \tau \in \mathrm{CS}(T_0) \\ C(\sigma\tau) \leq \pi}} \varepsilon(\tau) \right] = 0.$$

$\rightarrow$ only set-partitions π with type$(\pi) \preceq_d \lambda$ have a non-zero summand.
This proves triangularity. □

Proof of orthogonality

$$\langle t_\lambda, t_\mu \rangle = \sum_{\substack{\sigma \in \mathrm{RS}(T_\lambda) \\ \tau \in \mathrm{CS}(T_\lambda)}} \sum_{\substack{\sigma' \in \mathrm{RS}(T_\mu) \\ \tau' \in \mathrm{CS}(T_\mu)}} \varepsilon(\tau)\, \varepsilon(\tau')\, z_{\mathsf{type}(\sigma\tau)} [\mathsf{type}(\sigma\tau) = \mathsf{type}(\sigma'\tau')]$$

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Thus $\langle t_\lambda, t_\mu \rangle = 0$ unless $\mu \preceq_d \lambda$.
By symmetry $\langle t_\lambda, t_\mu \rangle = 0$ unless $\mu = \lambda$.
We have proved orthogonality. □

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Probably some nice combinatorial/algebraic framework hidden behind all this. . .