A bijection for rooted maps on general sufaces

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joint work with

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FPSAC 2015 - Daejeon, South Korea, 6th of July 2015.

I. Maps

Maps

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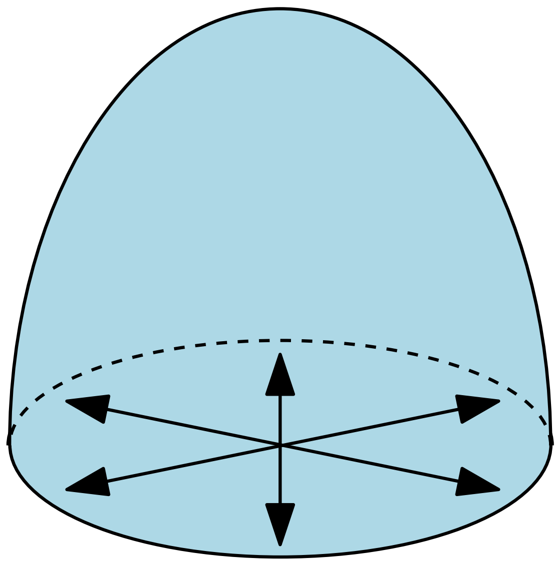
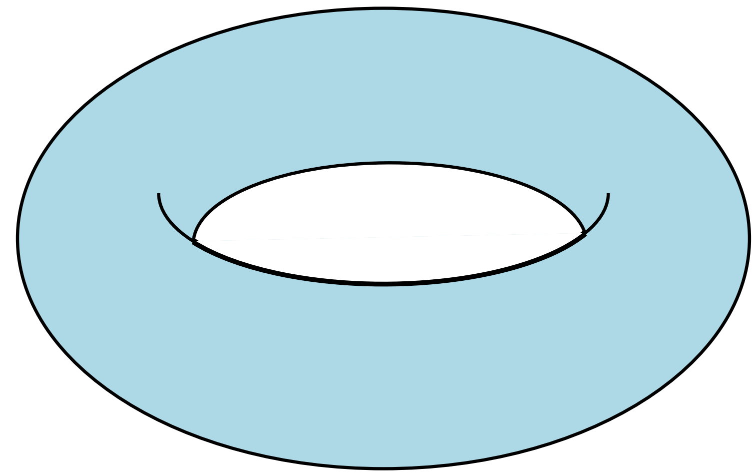


Projective plane



Torus

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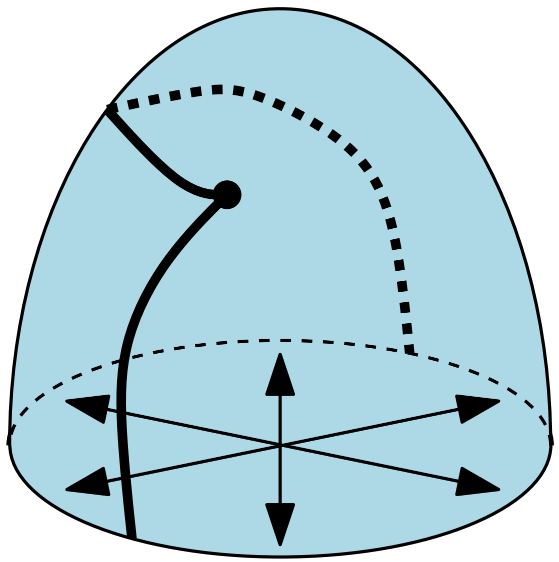
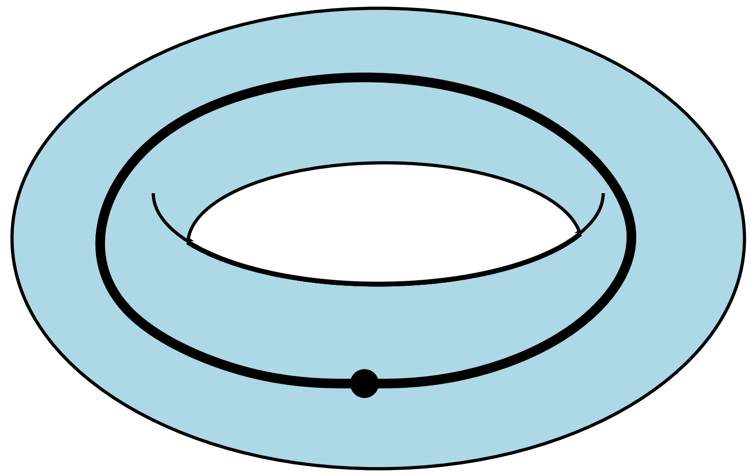


This is a map



This is not a map!

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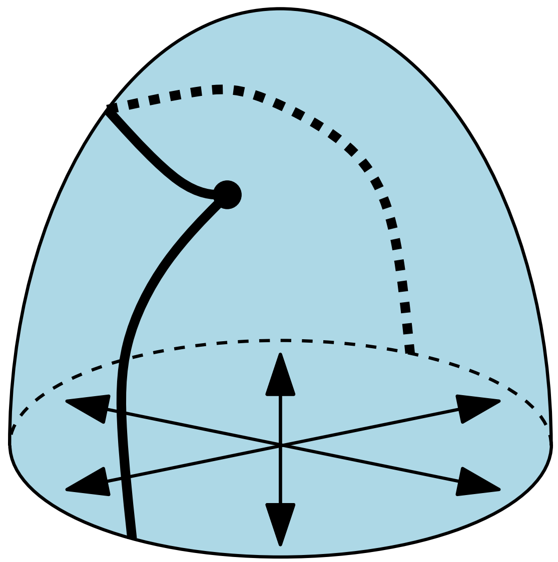

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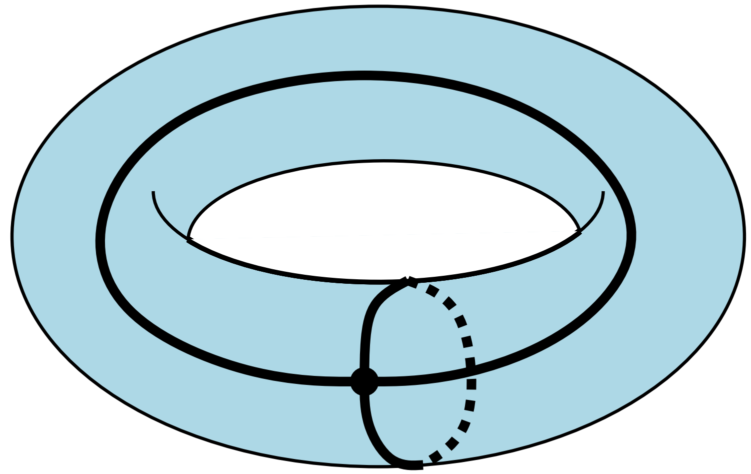

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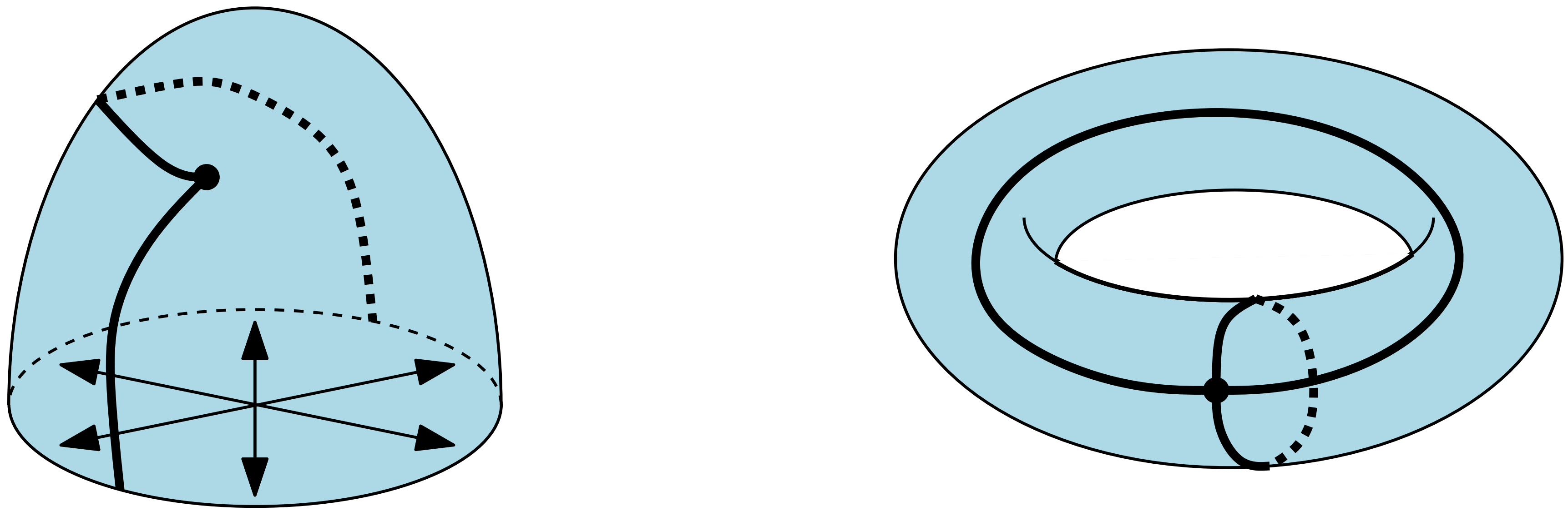

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Orientable vs. non-orientable

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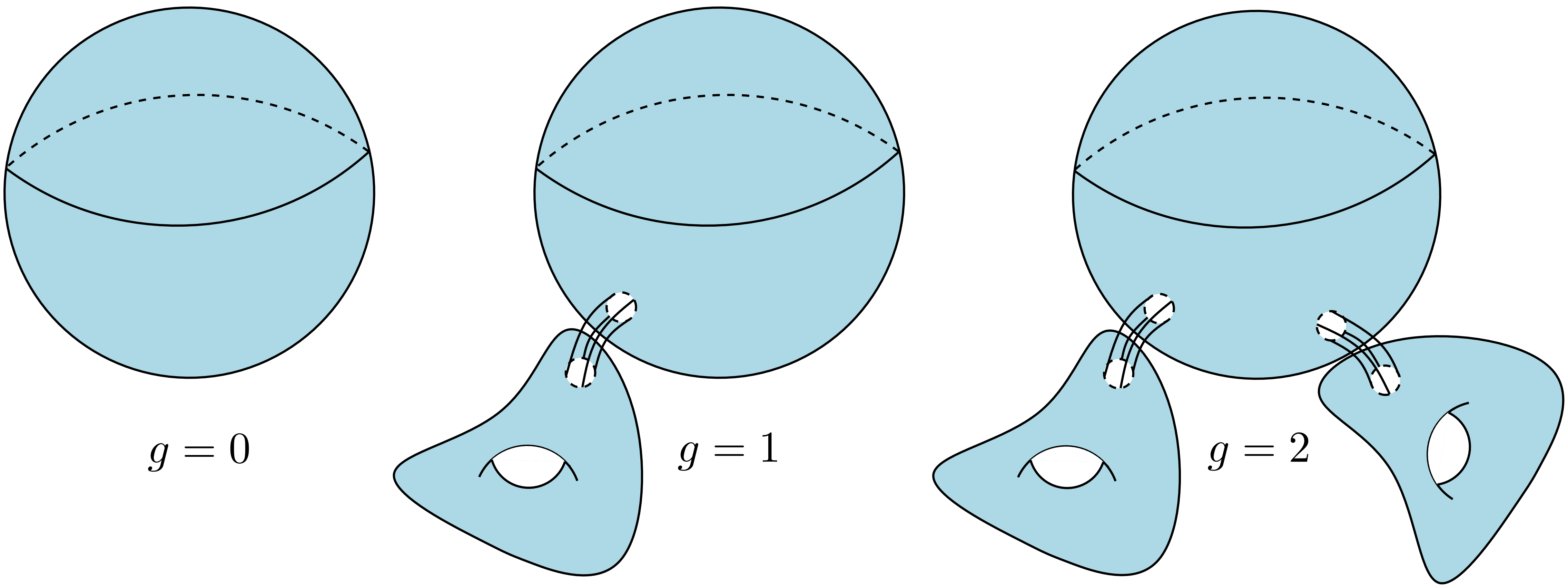

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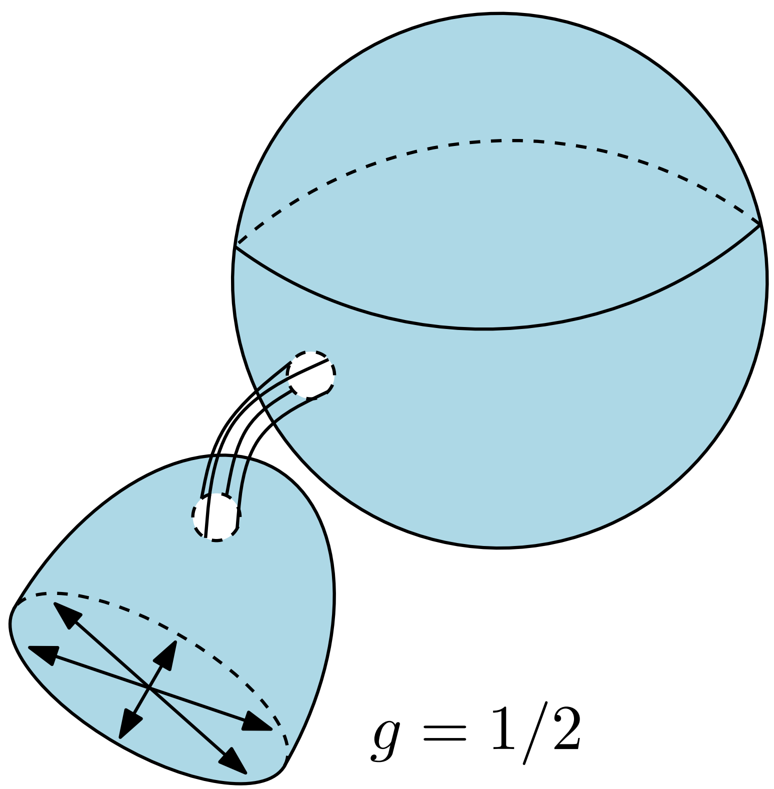

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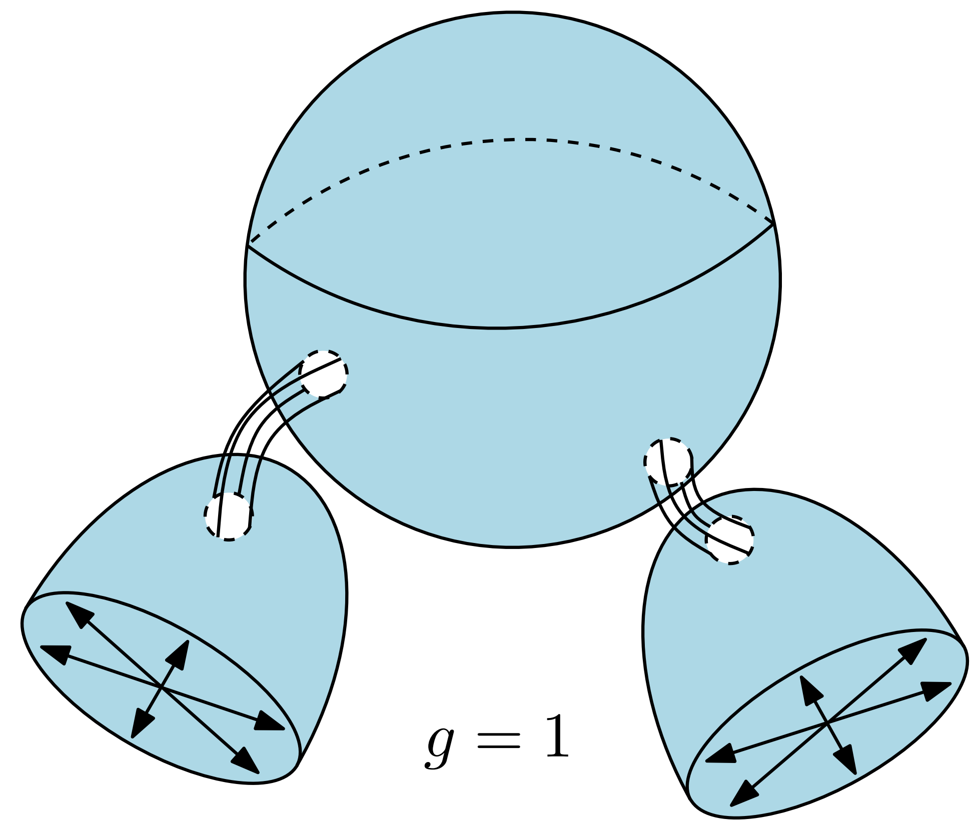

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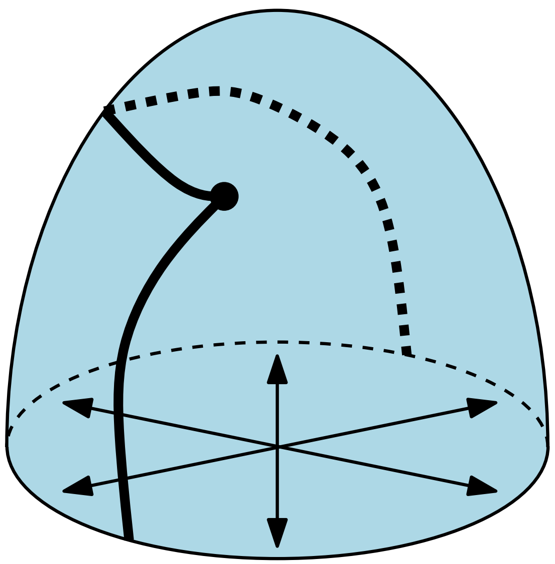
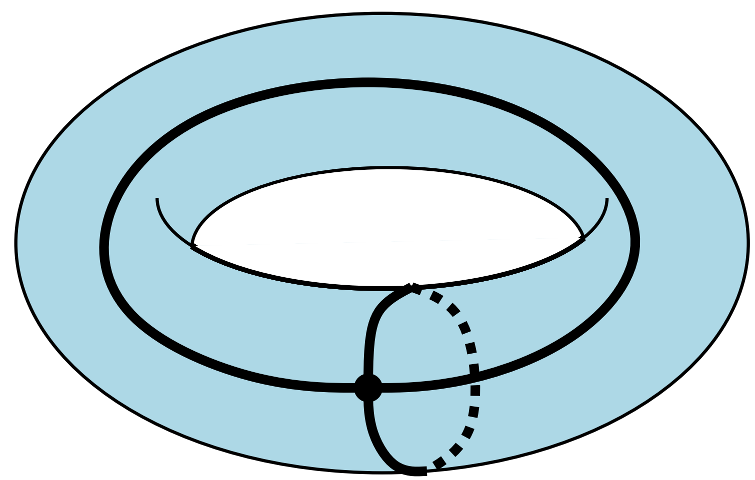
- orientable,
- non-orientable.

We will say that a map M is orientable/non-orientable of type g if the underlying surface is orientable/non-orientable of type g.



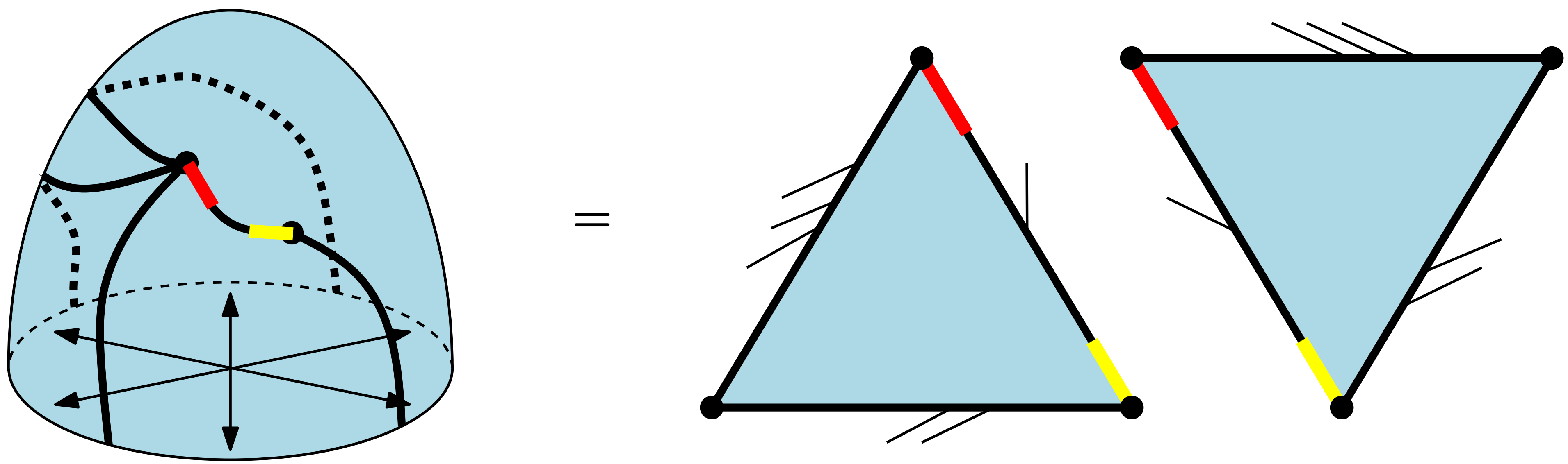
Non-orientable map of type $1/2$



Orientable map of type 1

Rooted maps

Each edge consists of two half-edges.



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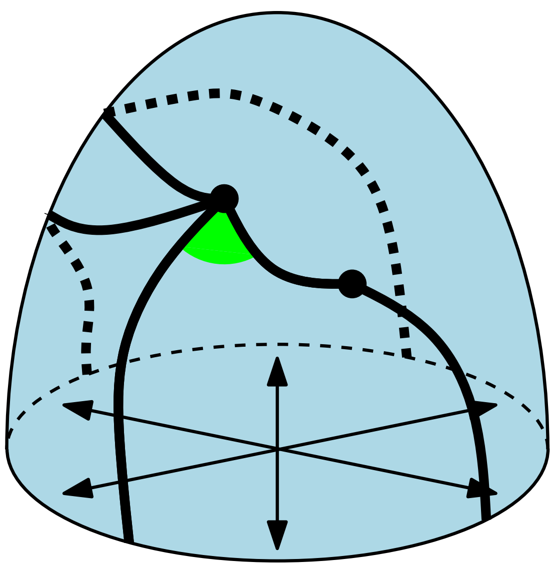

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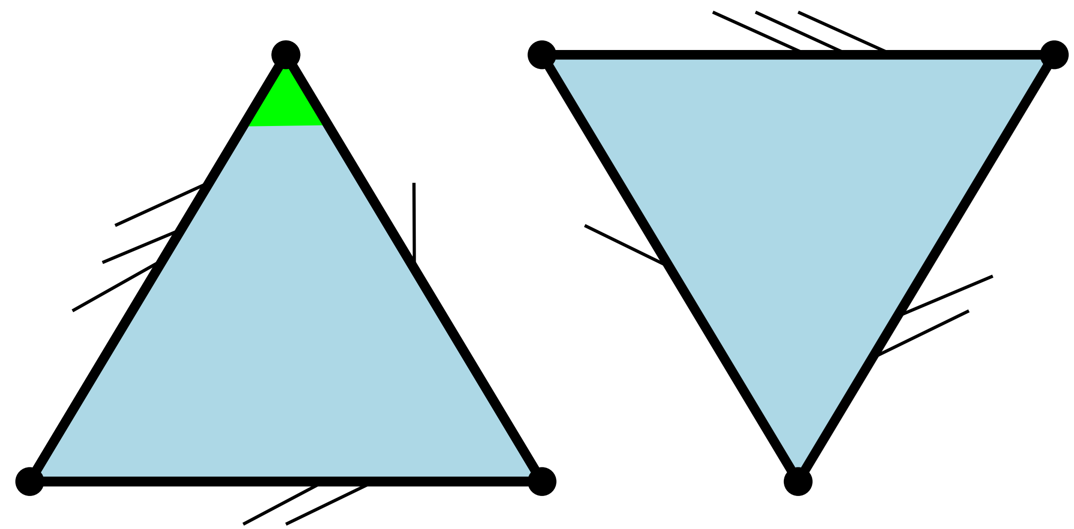

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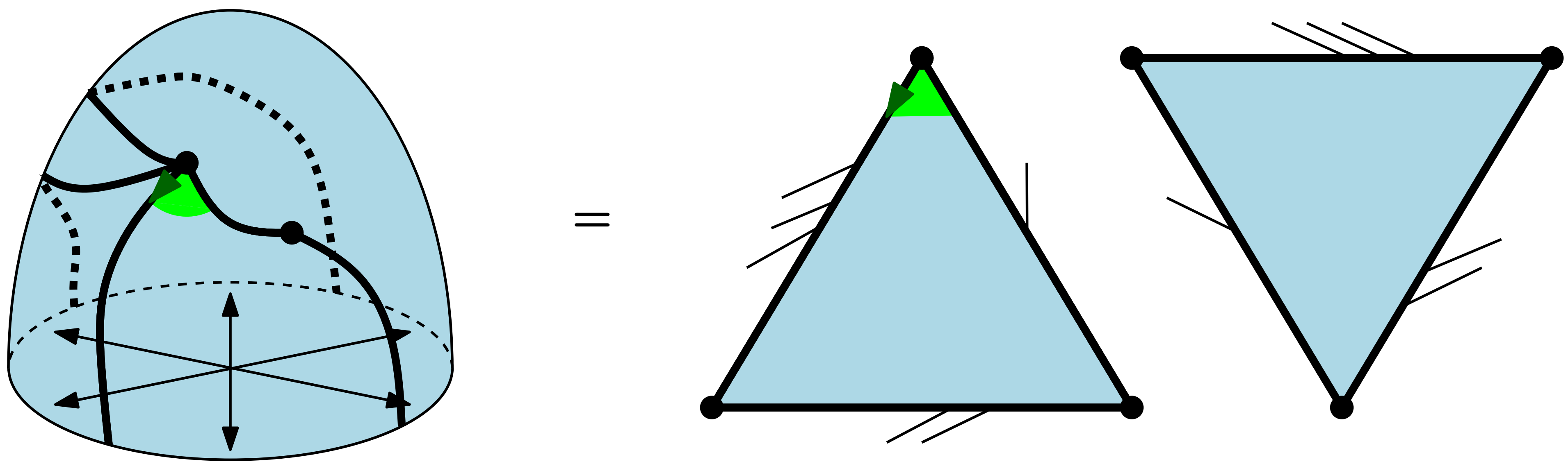

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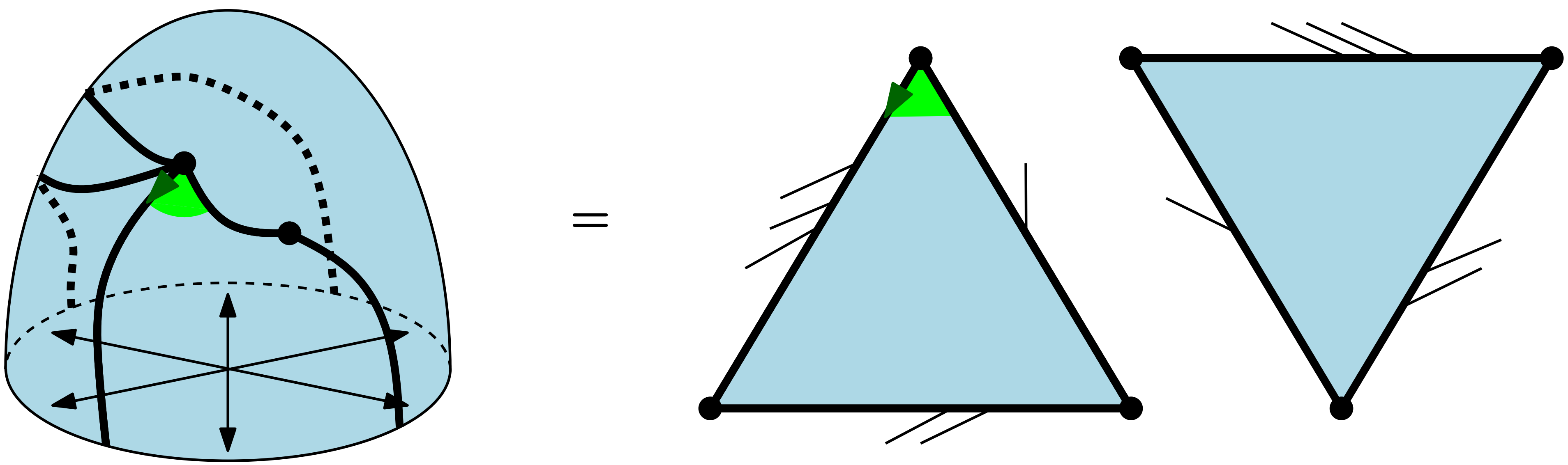

Remark:
Tutte noticed that maps are much simpler to enumerate, when rooted, because of the lack of symmetry. From now on, all maps will be rooted!

II. Enumeration of maps

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Combinatorial explanation:

- When $\mathbb{S}$ = sphere: bijection with labeled trees [Cori, Vauquelin 1981];
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- When $\chi(\mathbb{S}) = 2 - 2g$, and $\mathbb{S}$ is NON-ORIENTABLE: no combinatorial interpretation was known.

Maps with n edges vs. bipartite quadrangulations with n faces

Map M is bipartite if vertices can be colored by two different colors ($V_\bullet(M)$ - set of black vertices, $V_\circ(M)$ - set of white vertices, root vertex is black by convention) such that each edge connects two vertices of different colors.

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There is a bijection between

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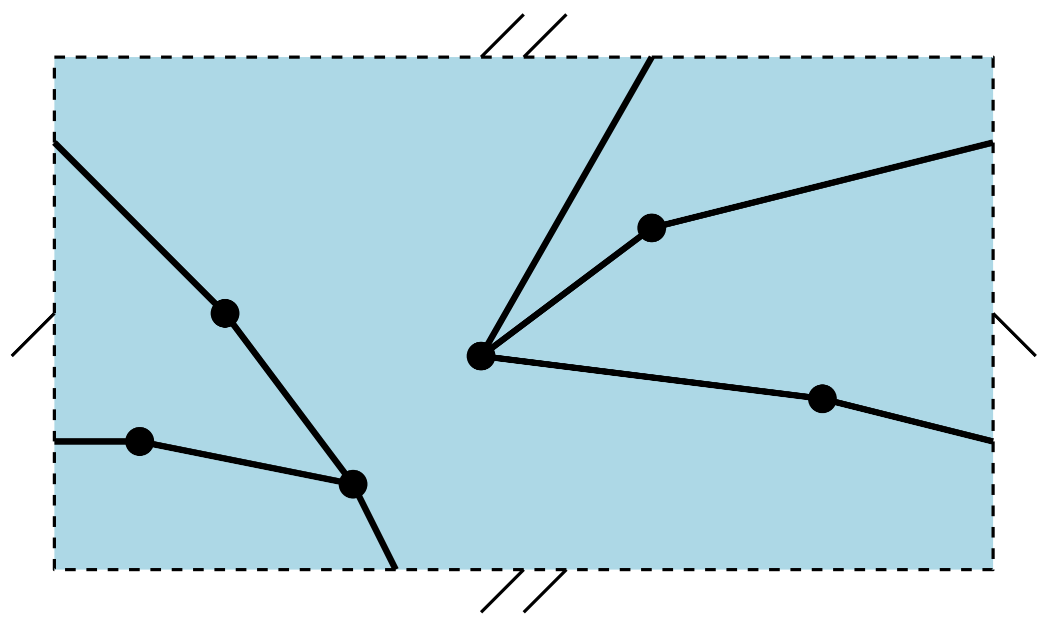

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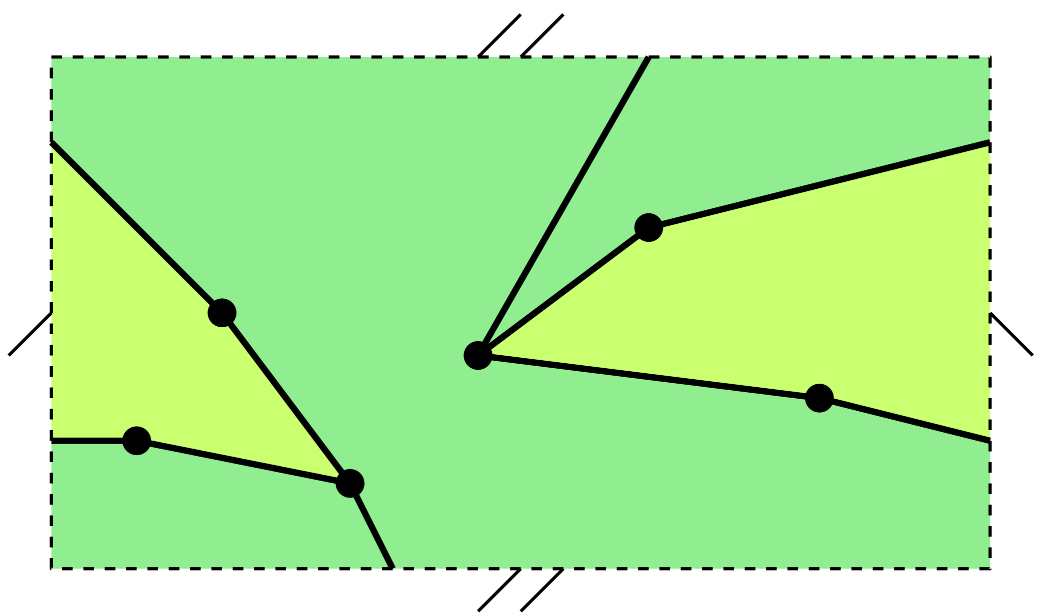

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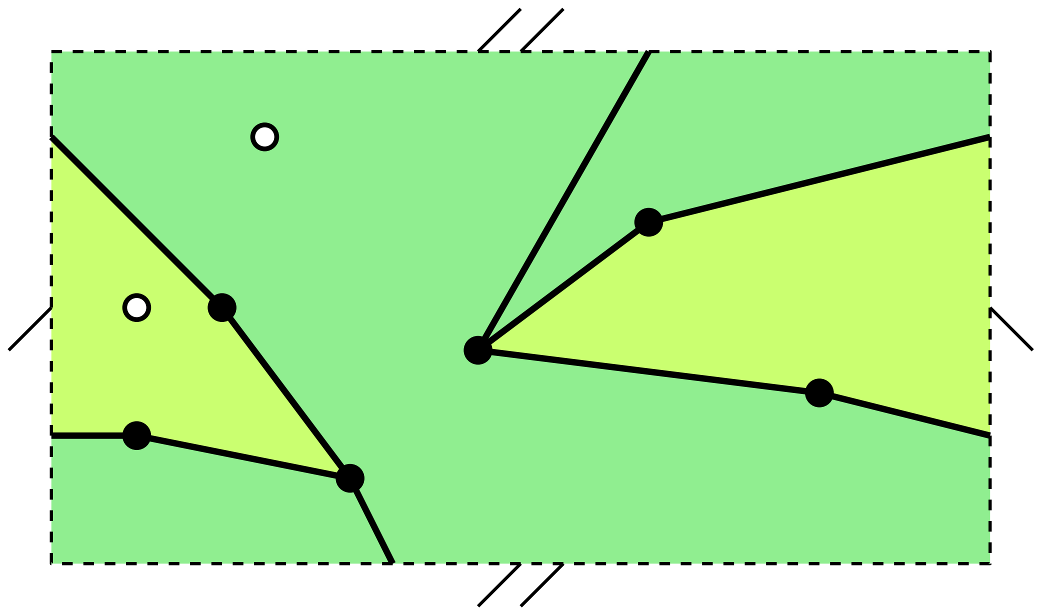

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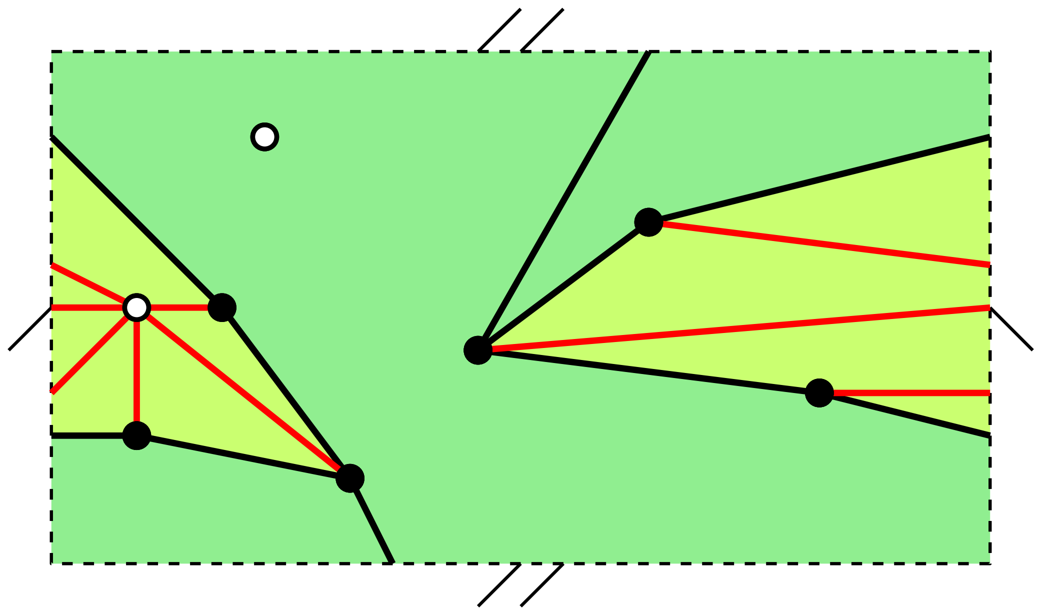

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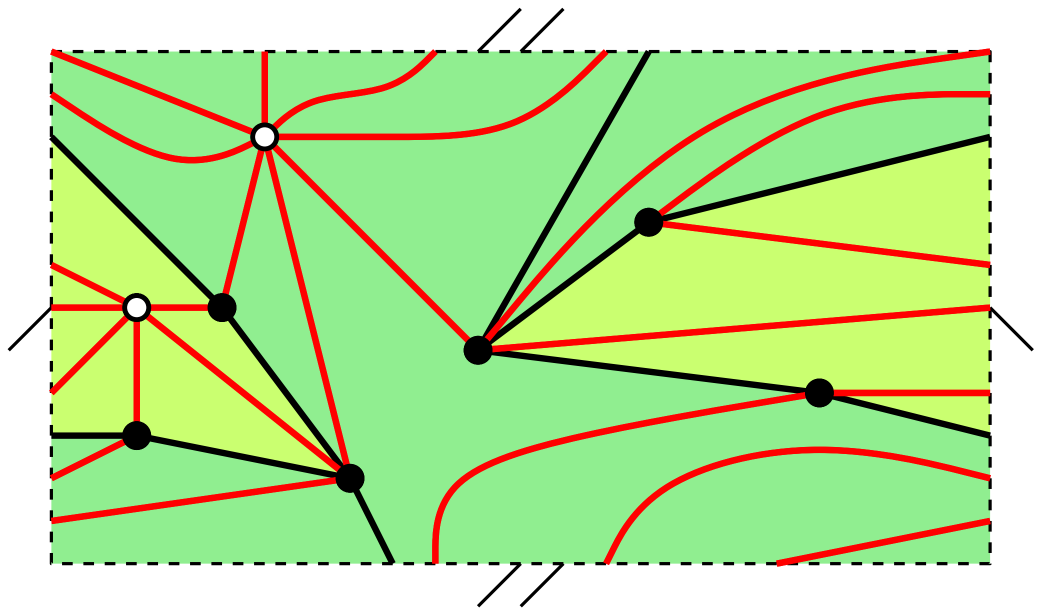

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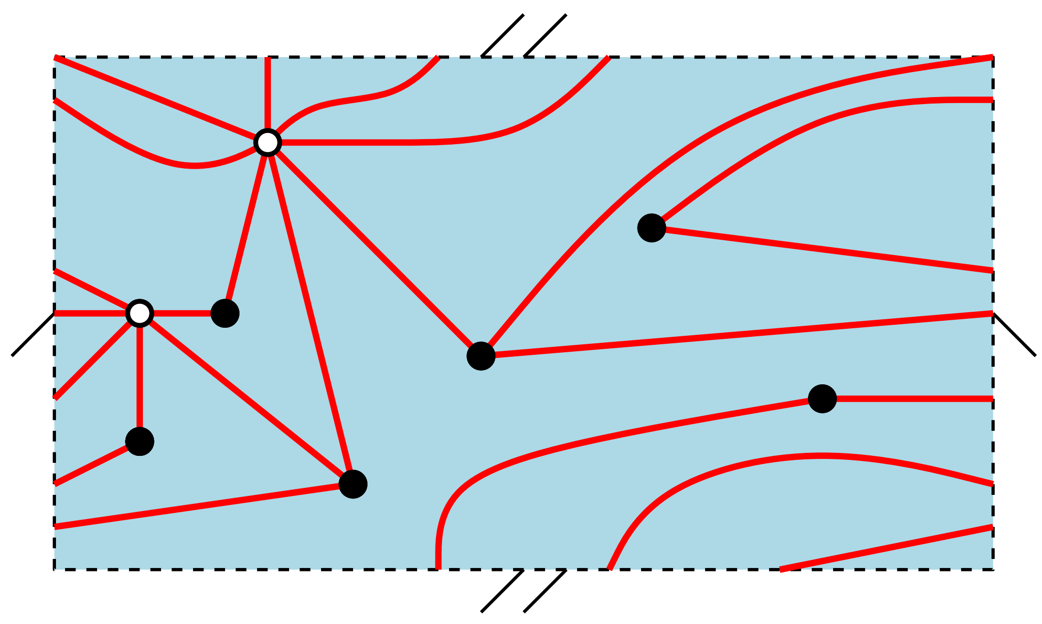

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Number of maps with n edges on $\mathbb{S}$	$=$	Number of bipartite quadrangulations with n faces on $\mathbb{S}$

Labeled and well-labeled maps

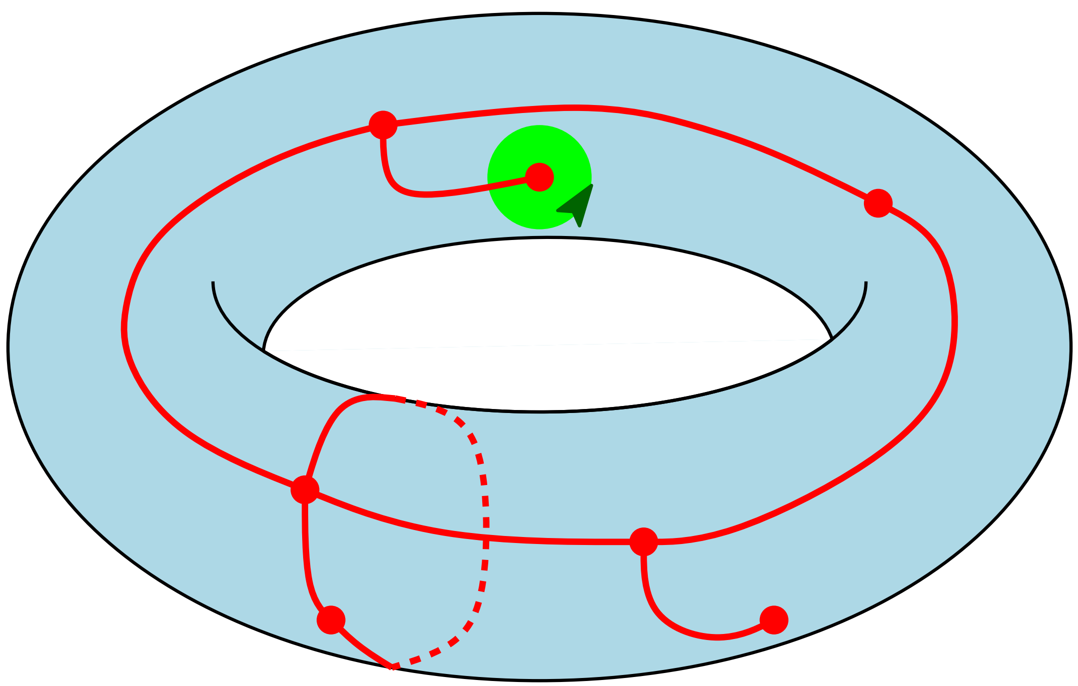
A map is called labeled if its vertices are labeled by integers such that:

- the root vertex has label 1;
- if two vertices are linked by an edge, their labels differ by at most 1.

If in addition we have:

- all the vertex labels are positive,

then the map is called well-labeled.



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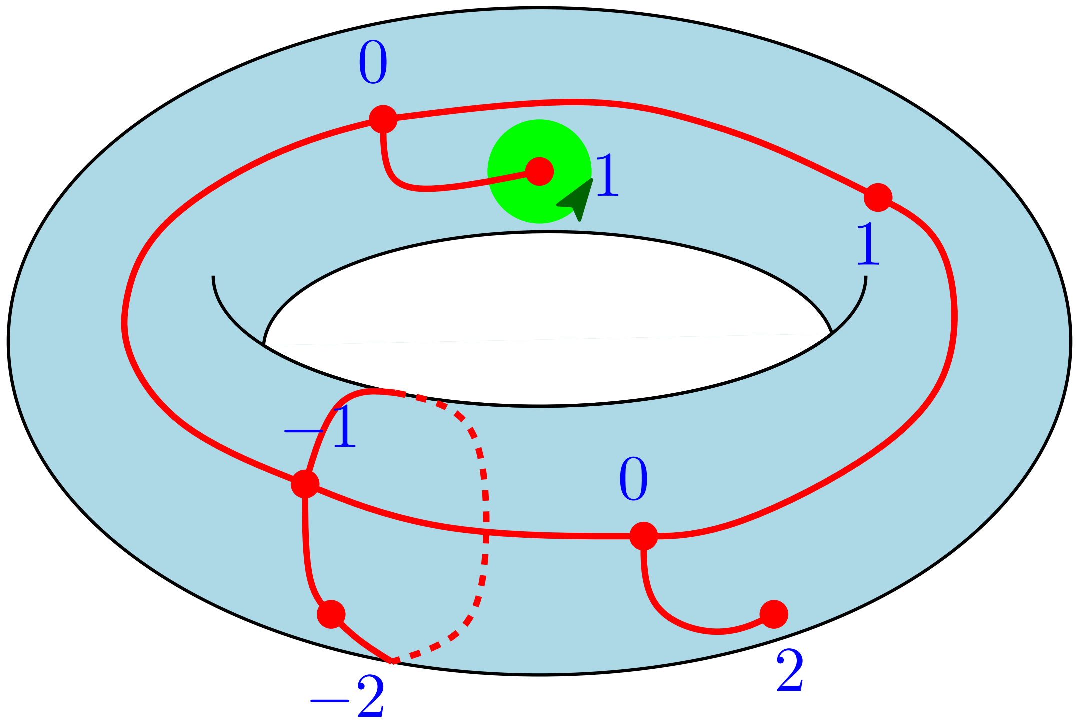

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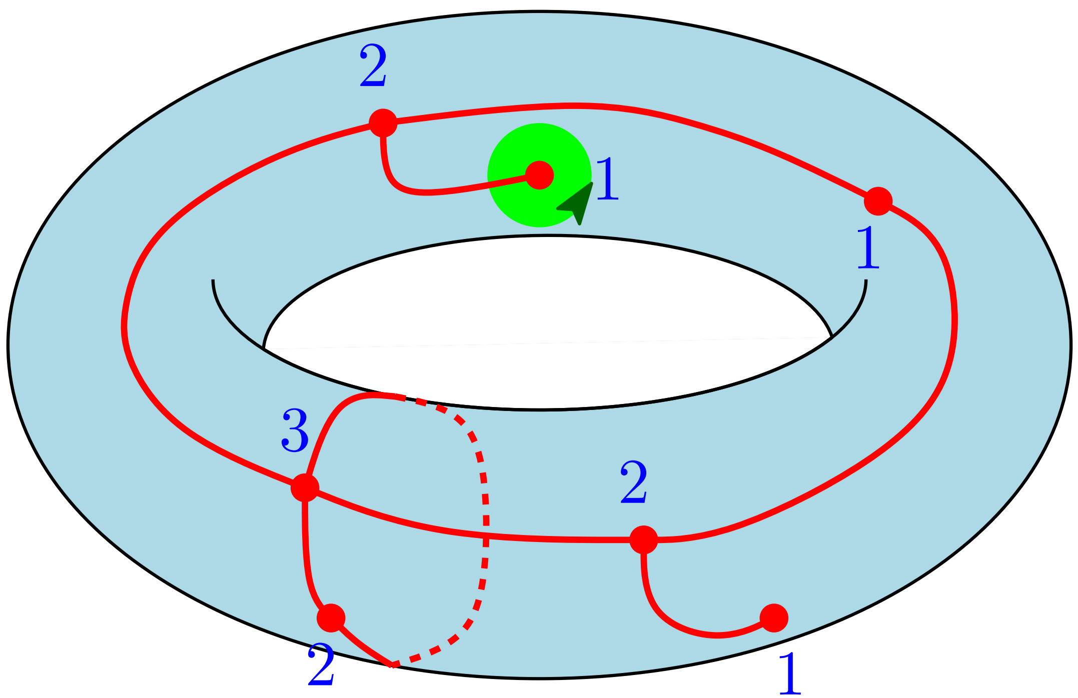

this map is labeled and well-labeled as well

Orientable case

Theorem [Marcus, Schaeffer 1996]
There exists a bijection between:

- rooted, bipartite quadrangulations on ORIENTABLE surface $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$;
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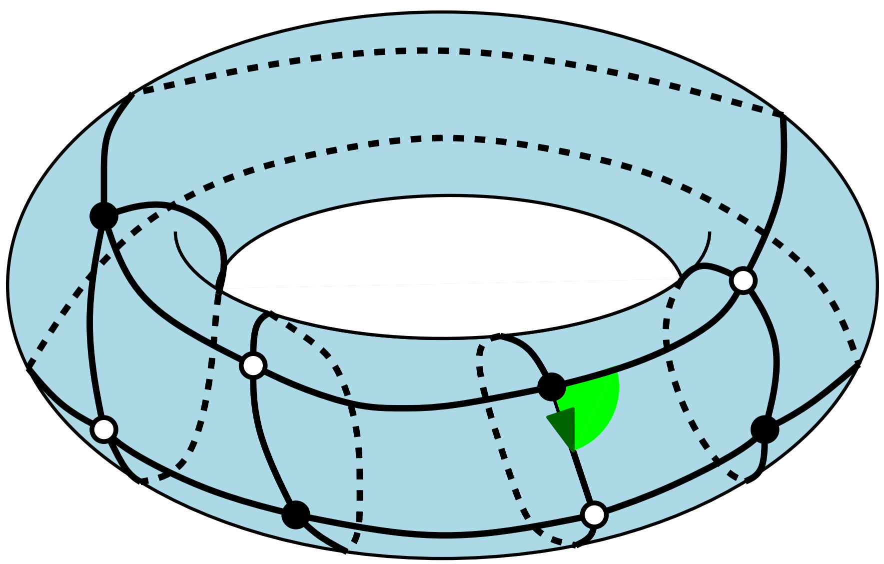

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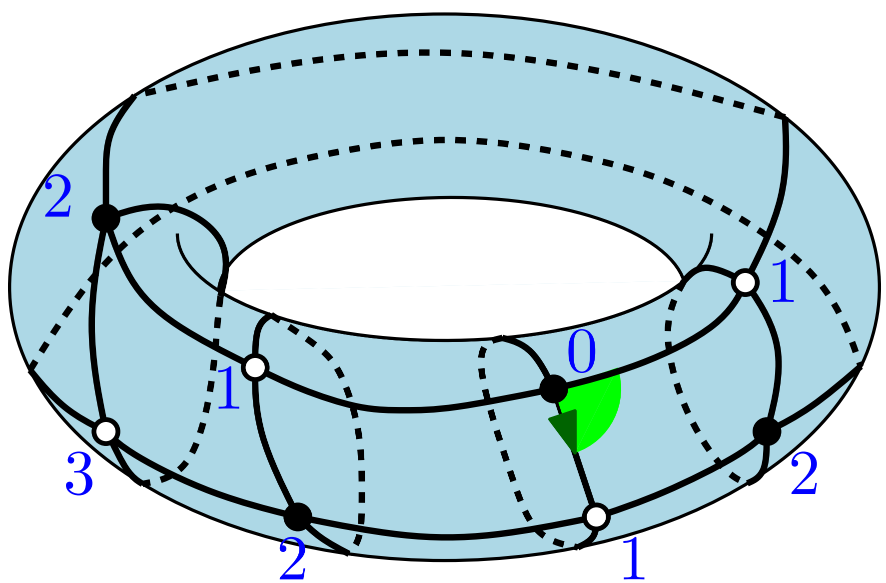

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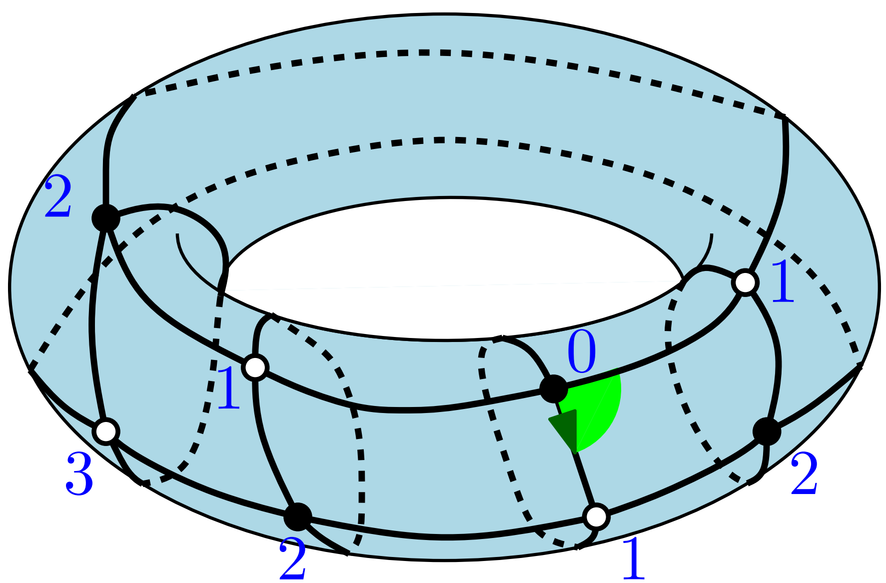

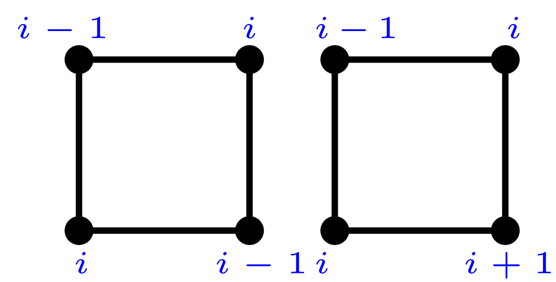

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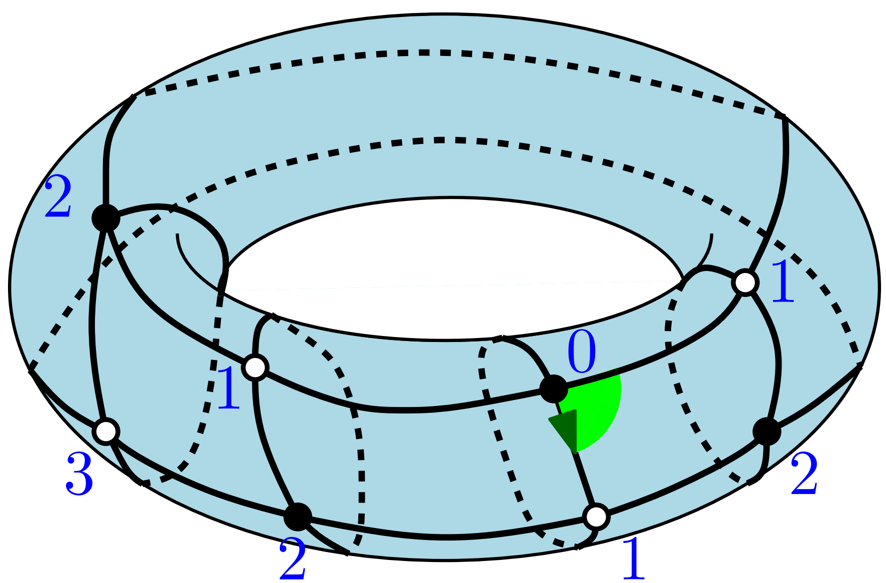

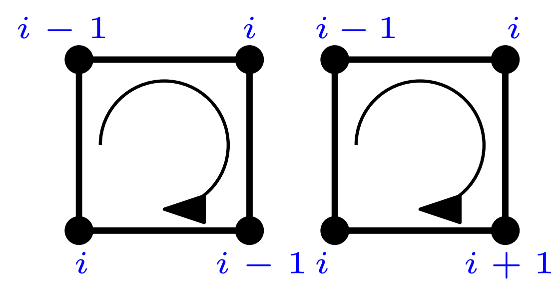

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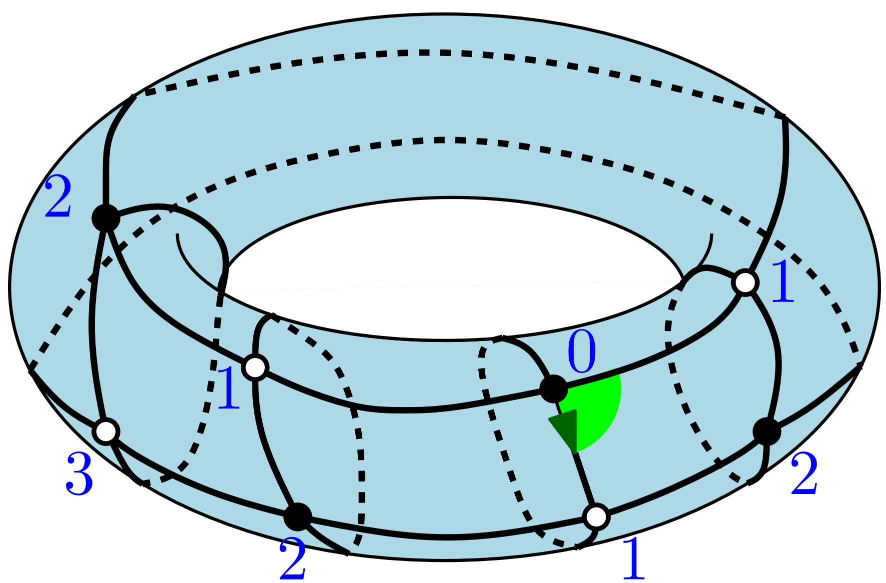

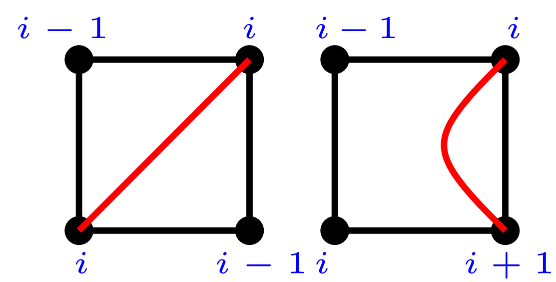

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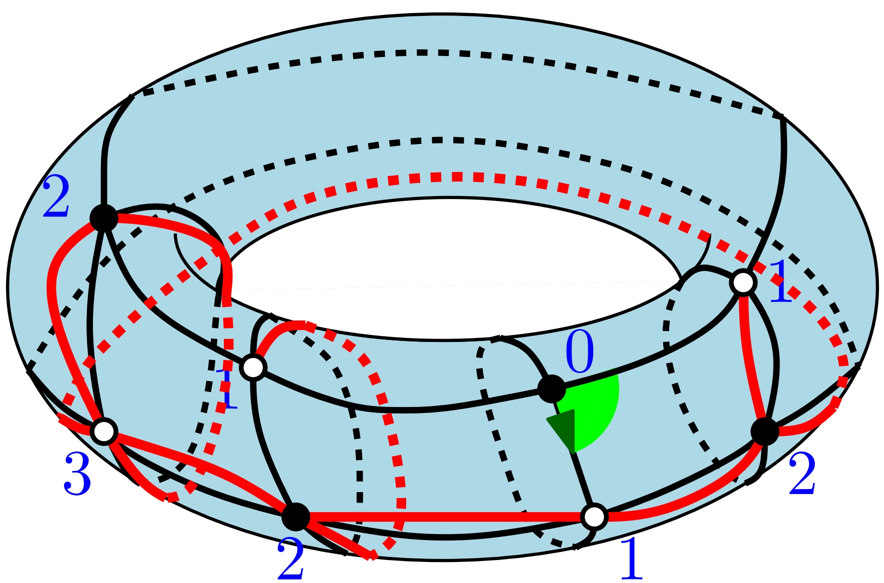

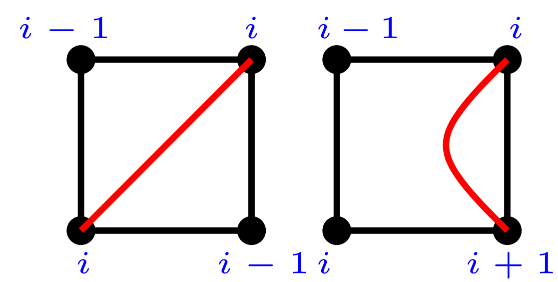

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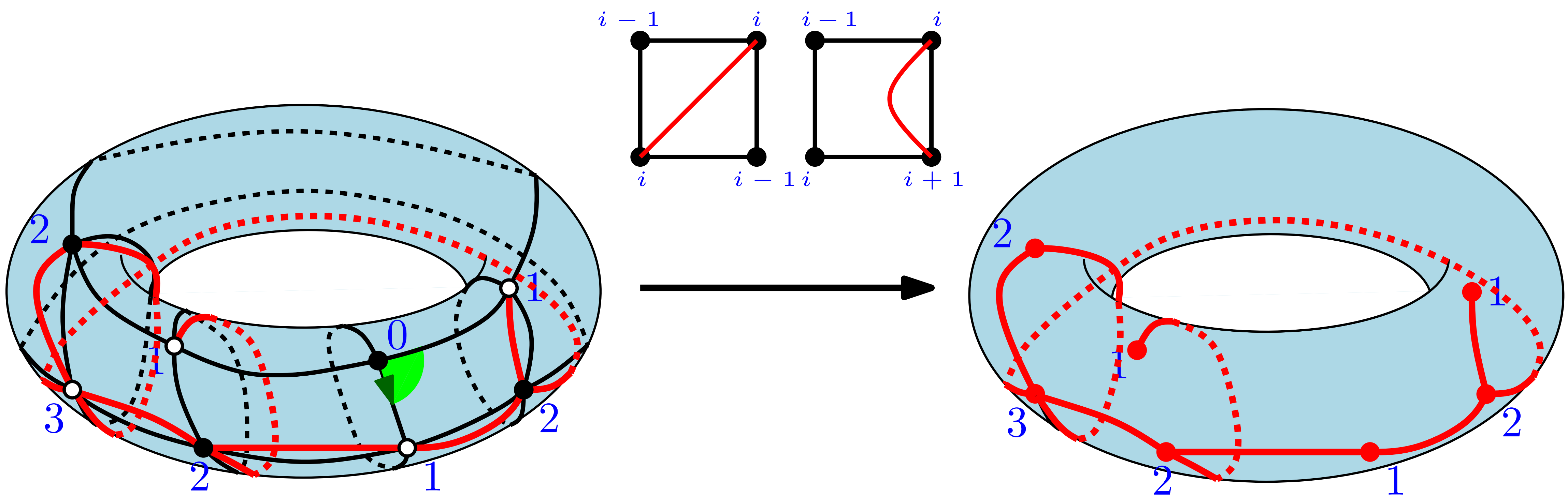

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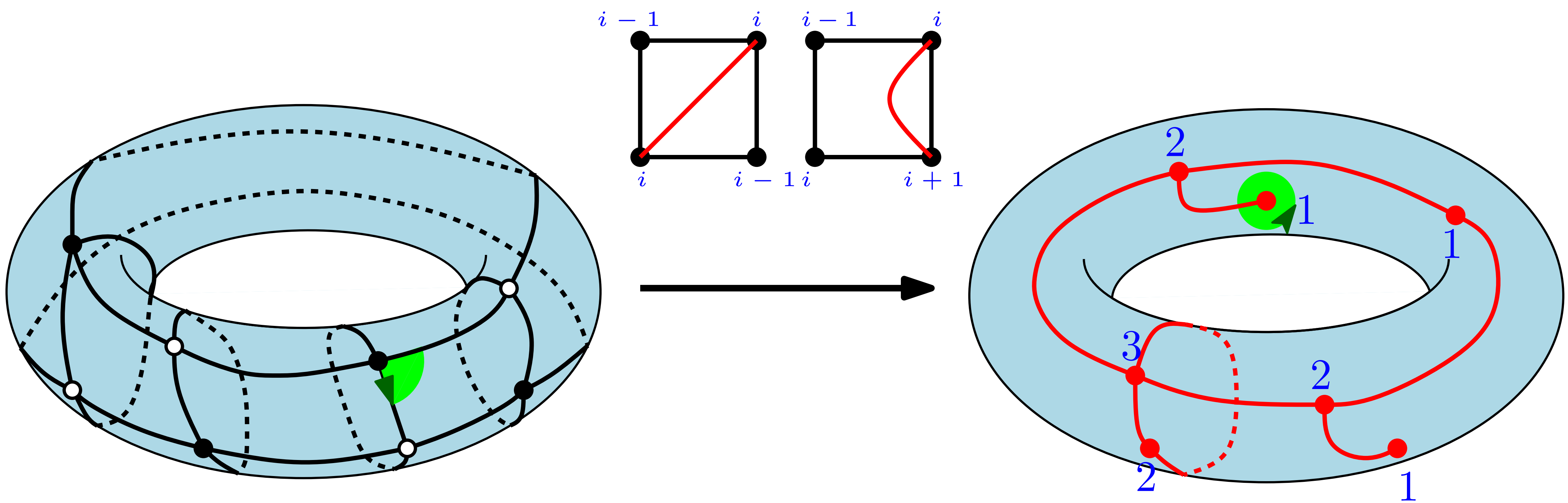

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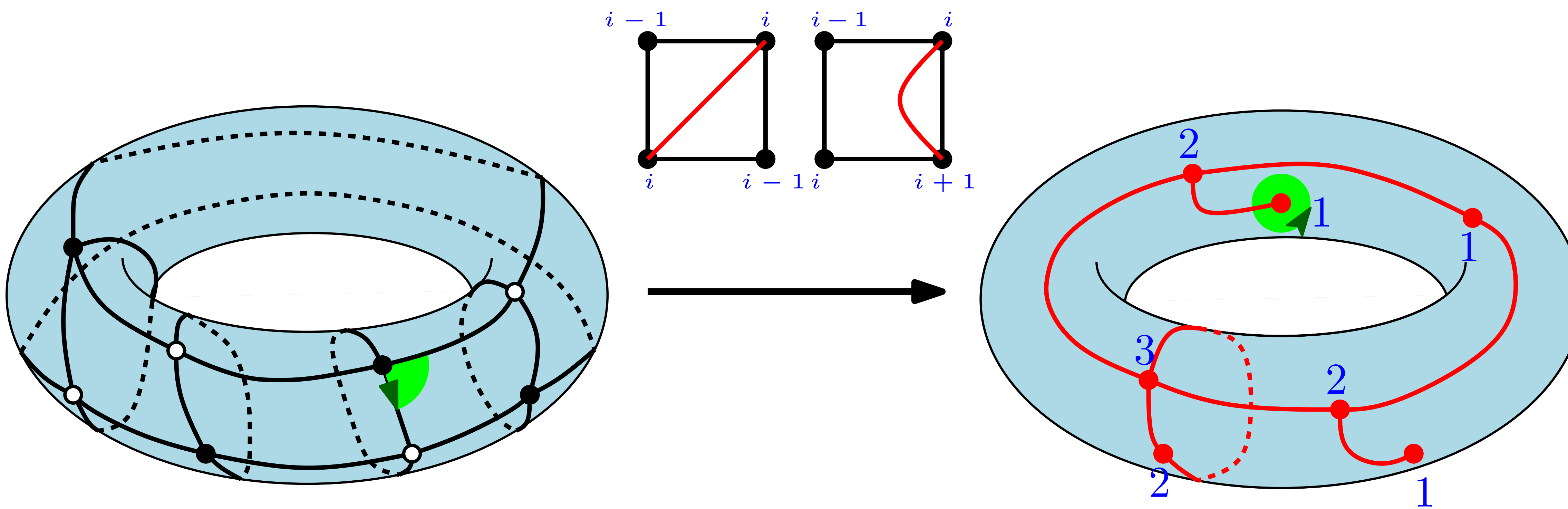

Are **non-orientable** maps different?

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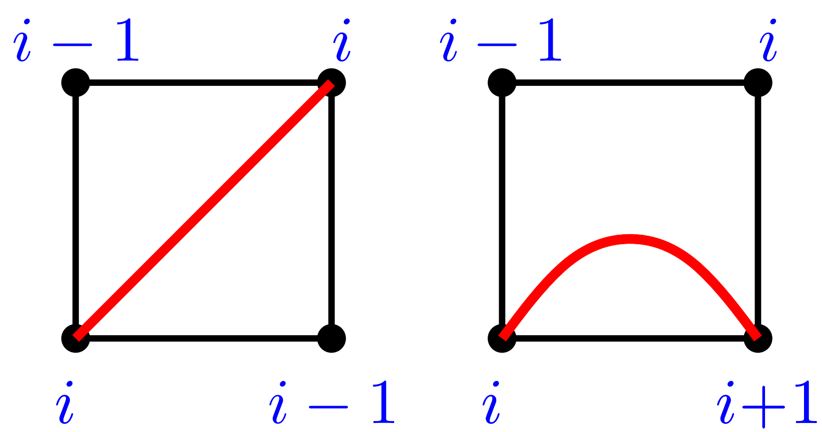

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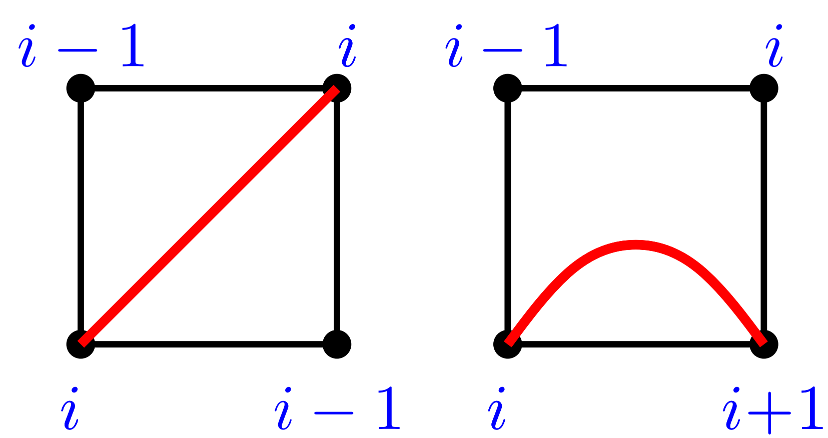

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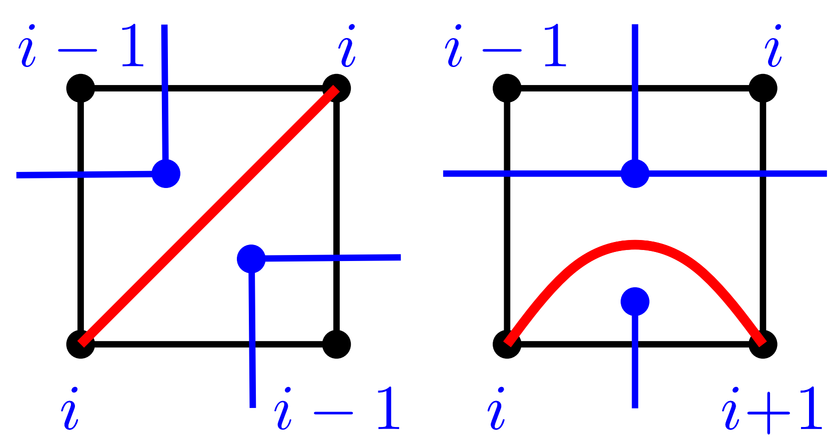

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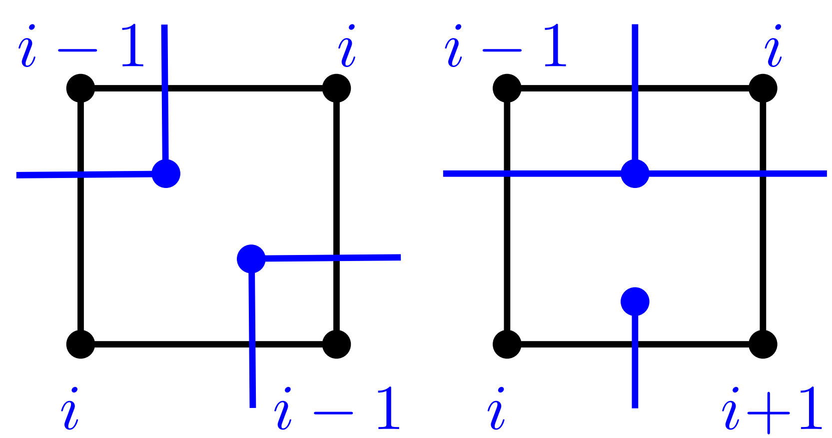

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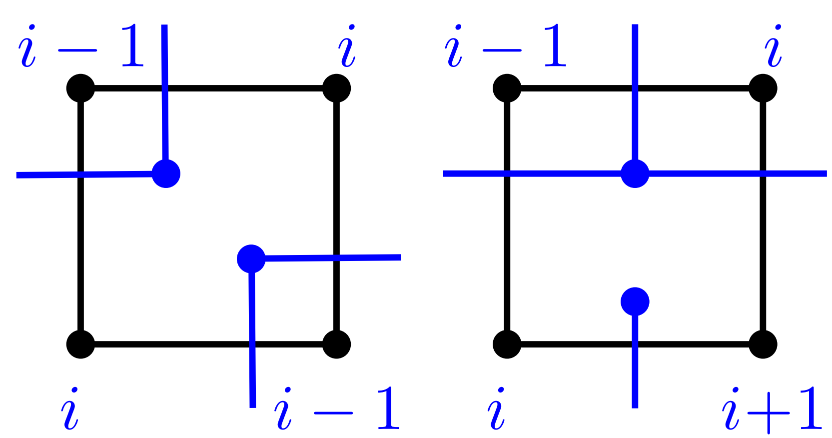

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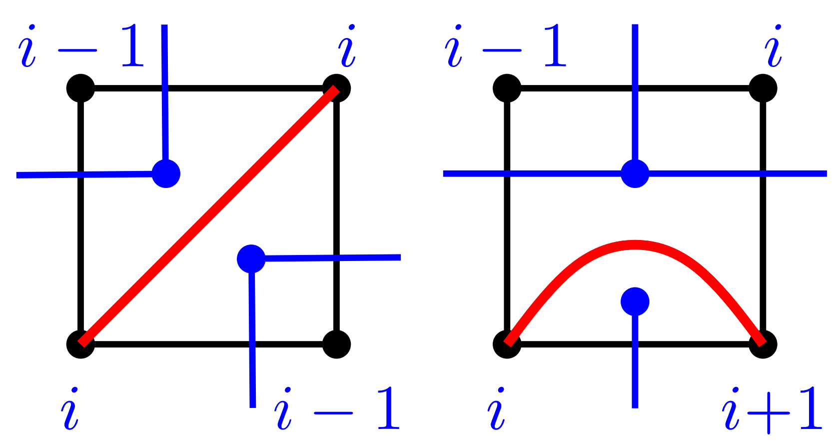

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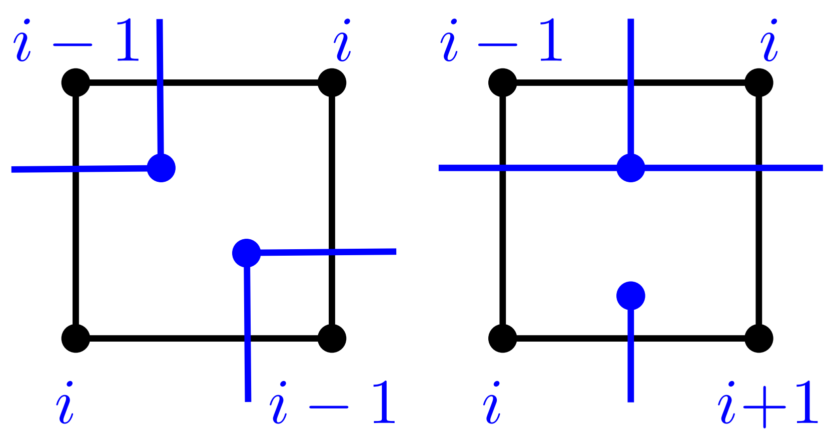
• If the construction of blue graph is local then it is invertible and it leads to a **BIJECTION!**



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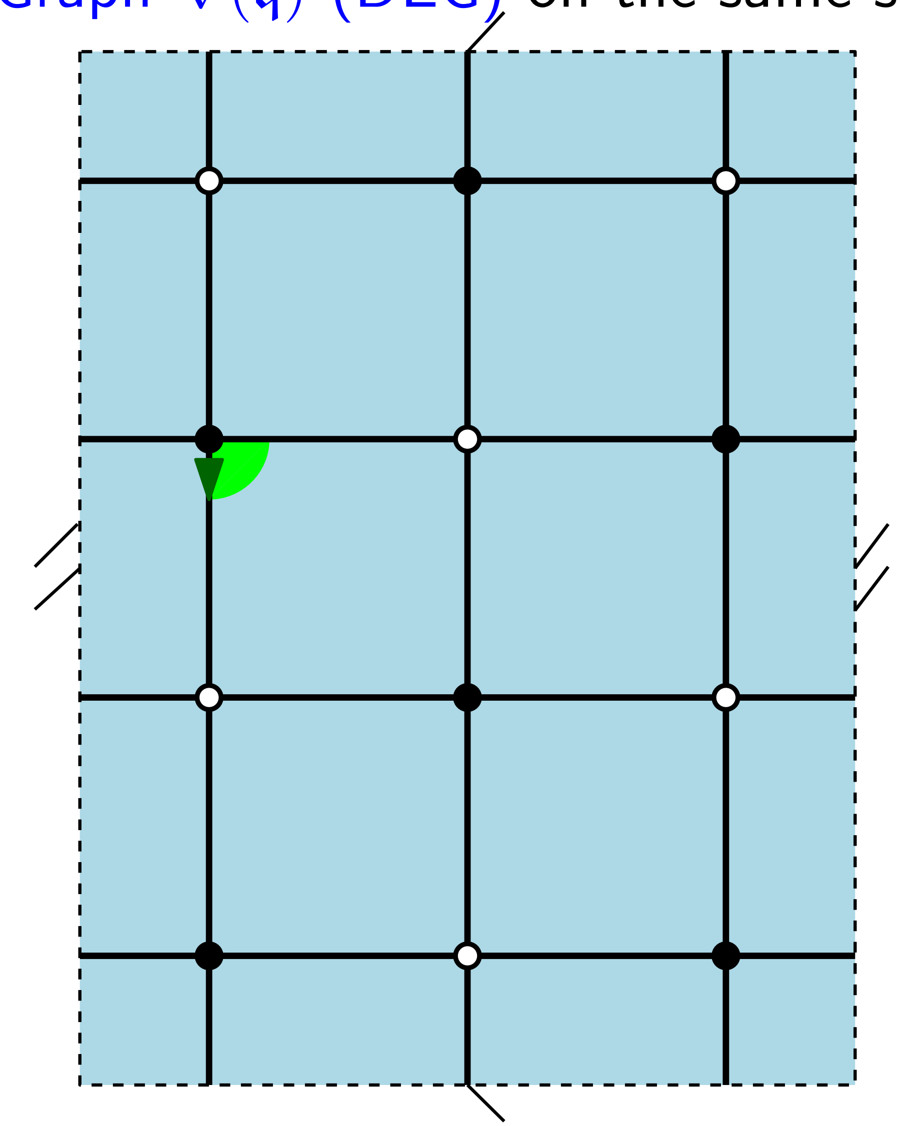

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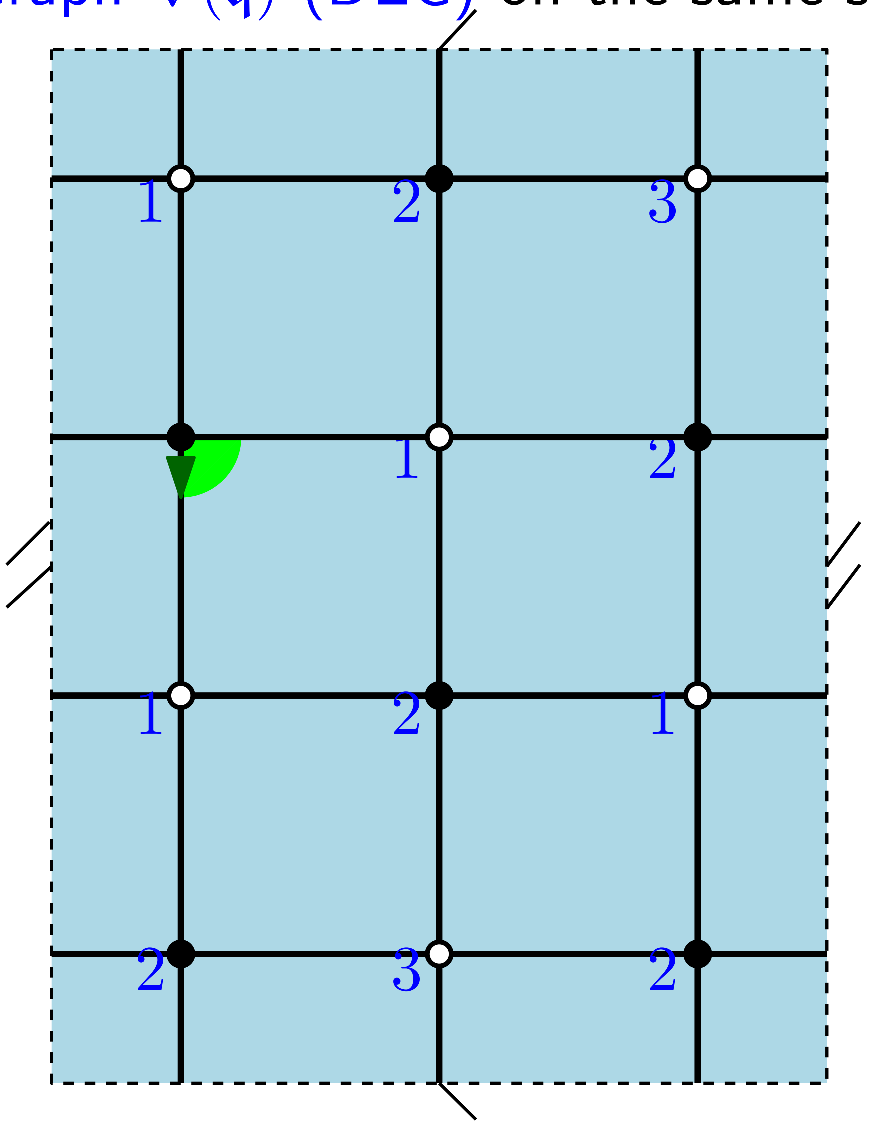

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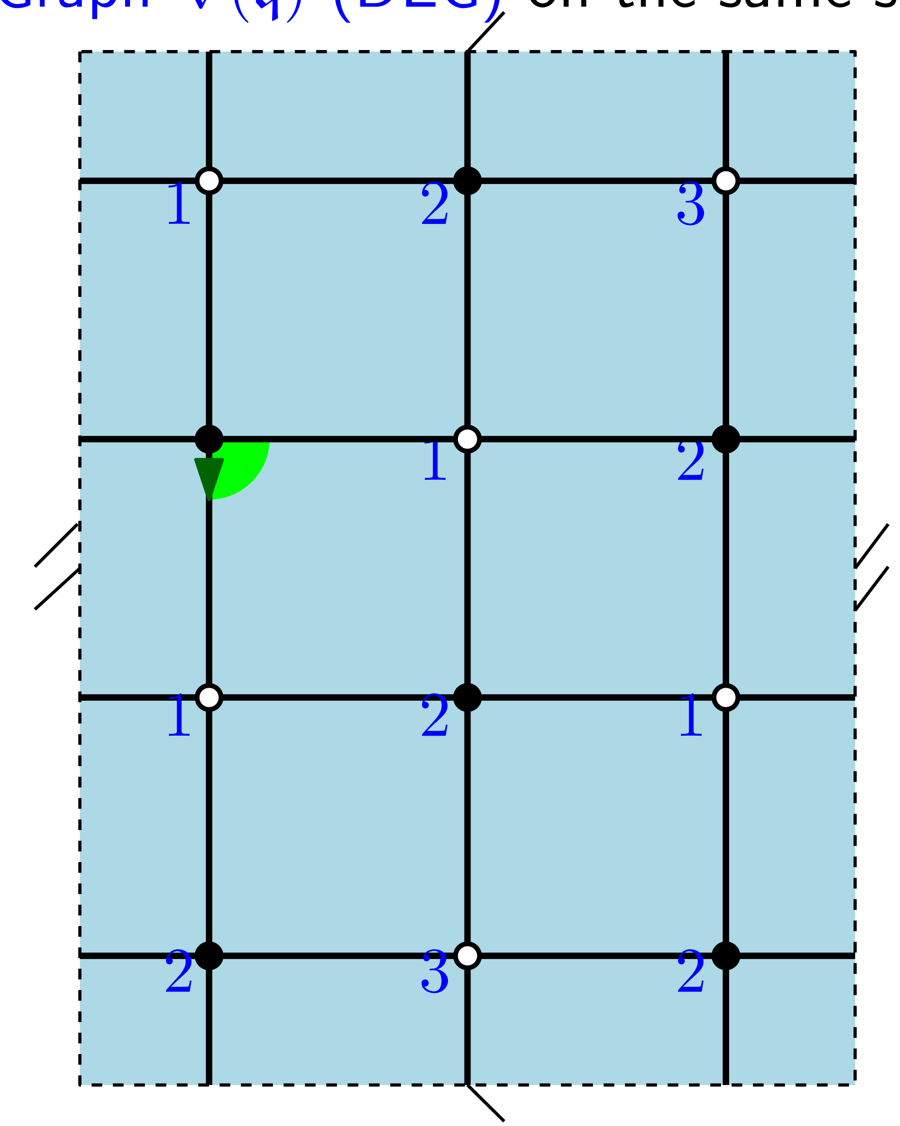
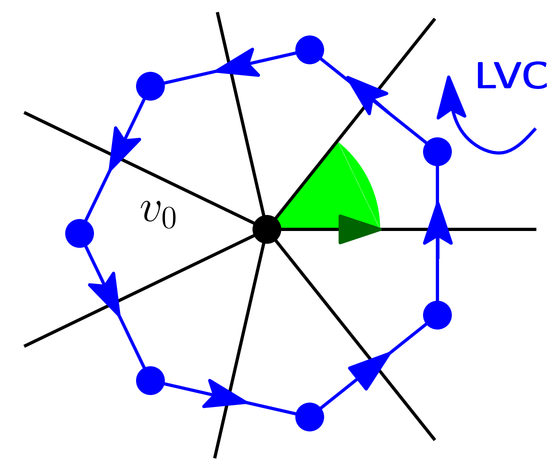


Step 0: Initialization



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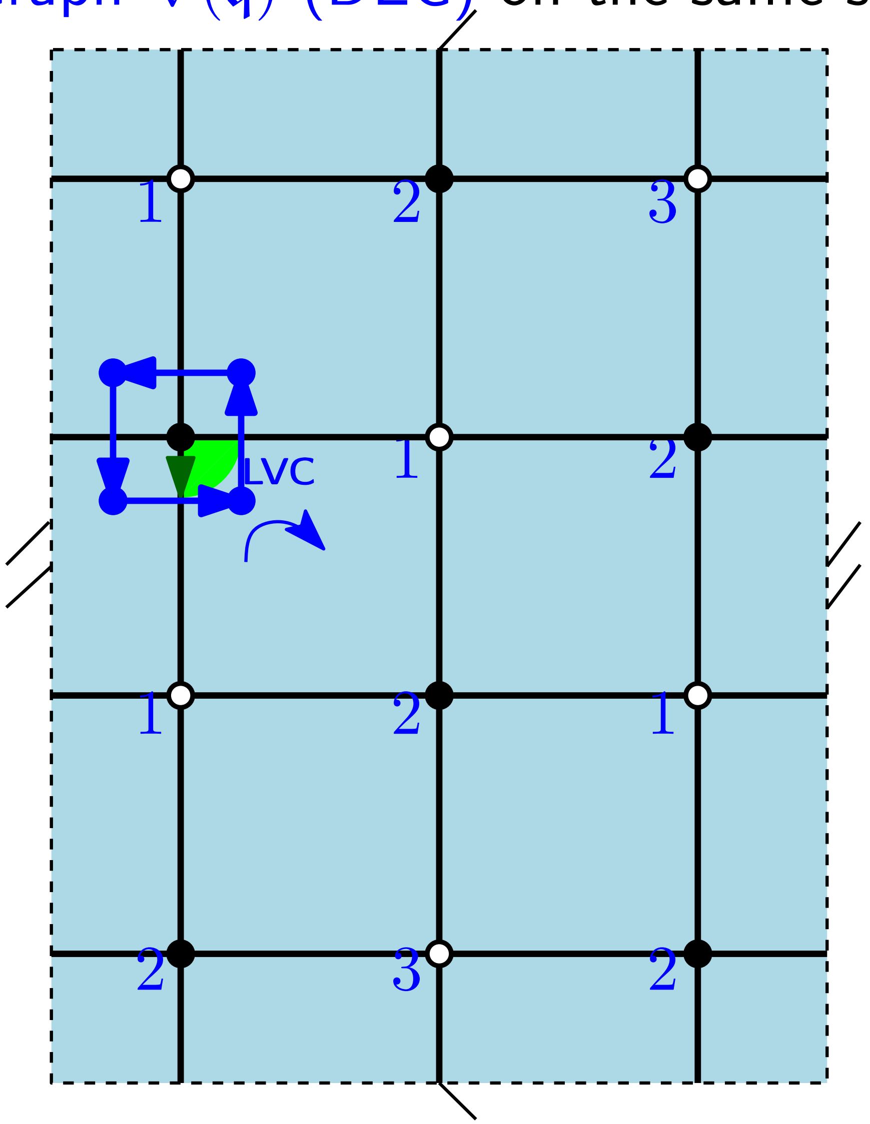

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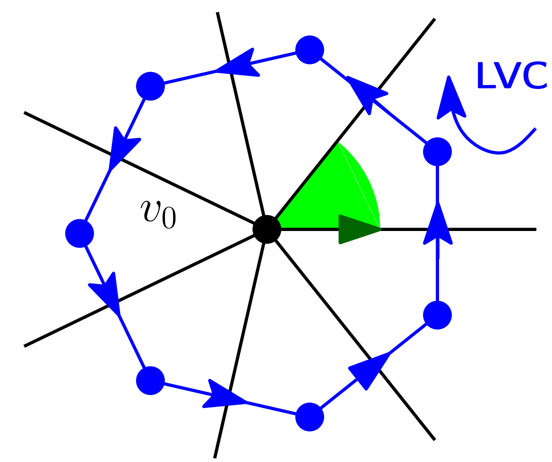

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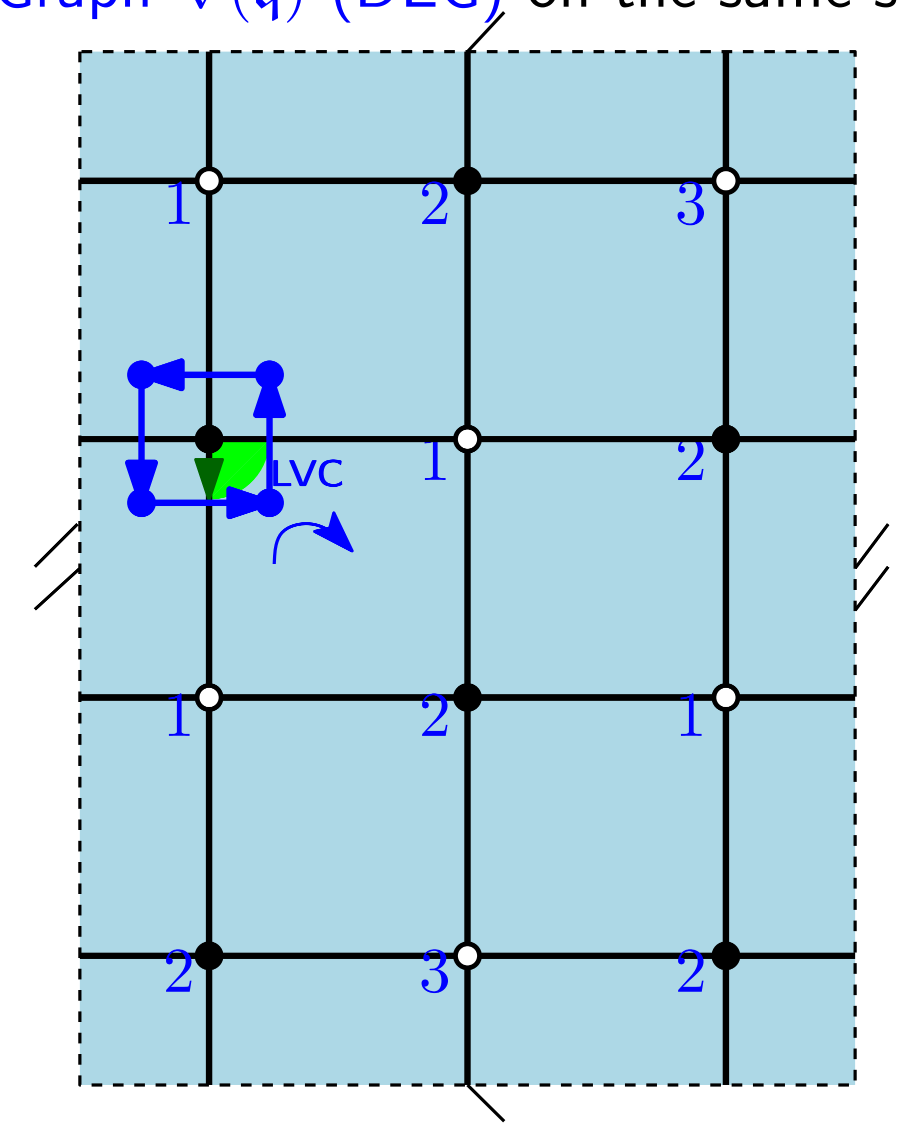

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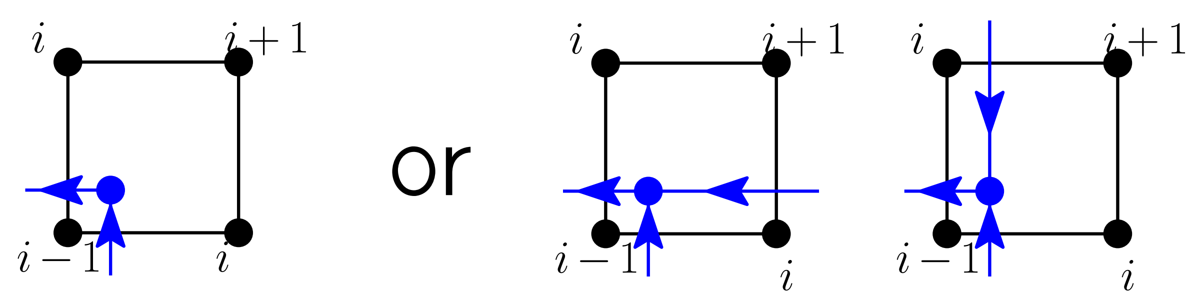

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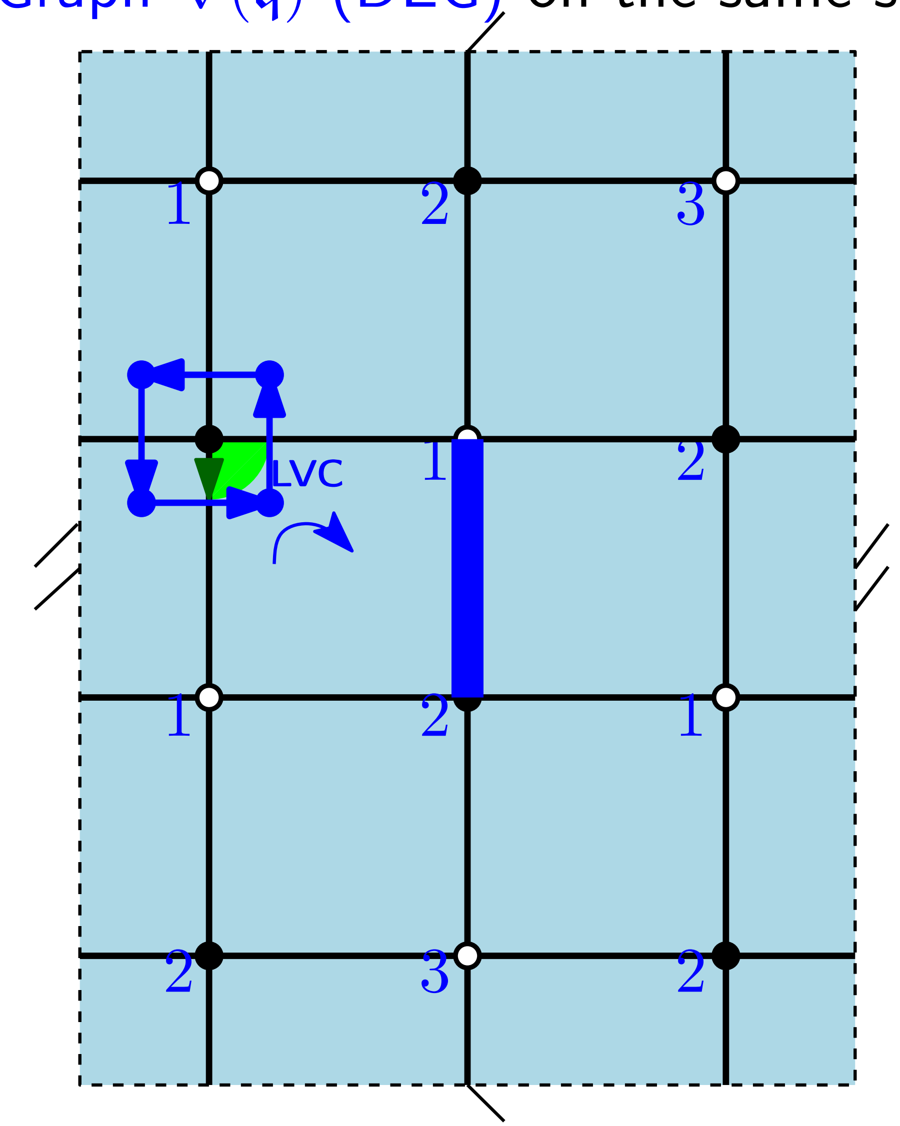

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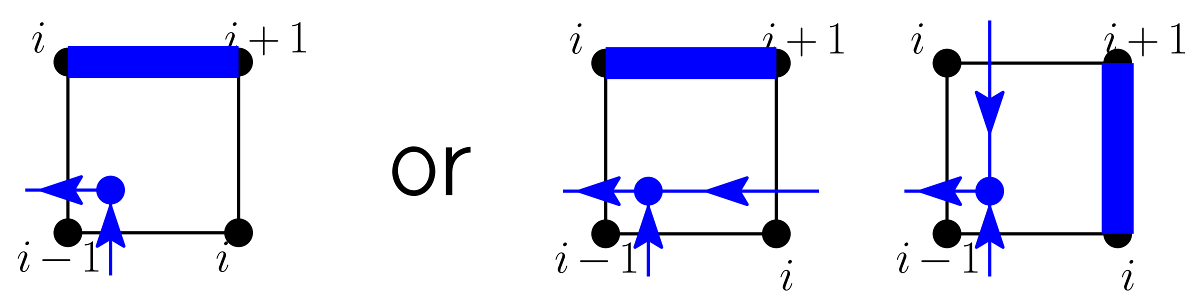

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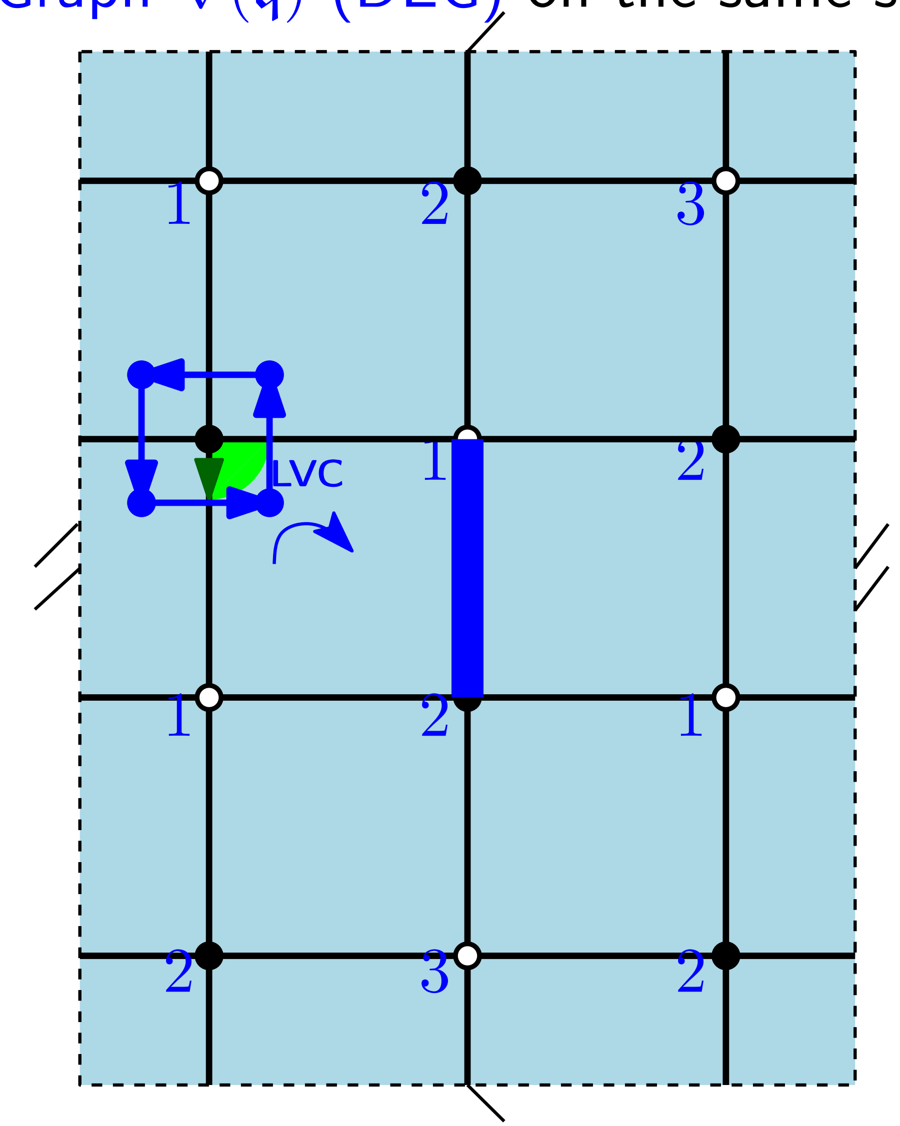

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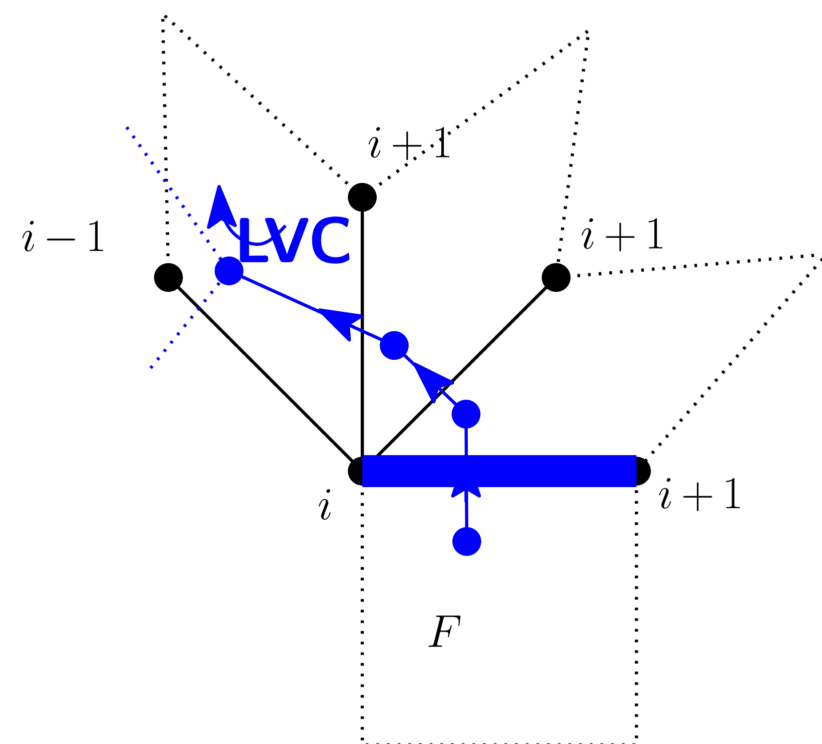

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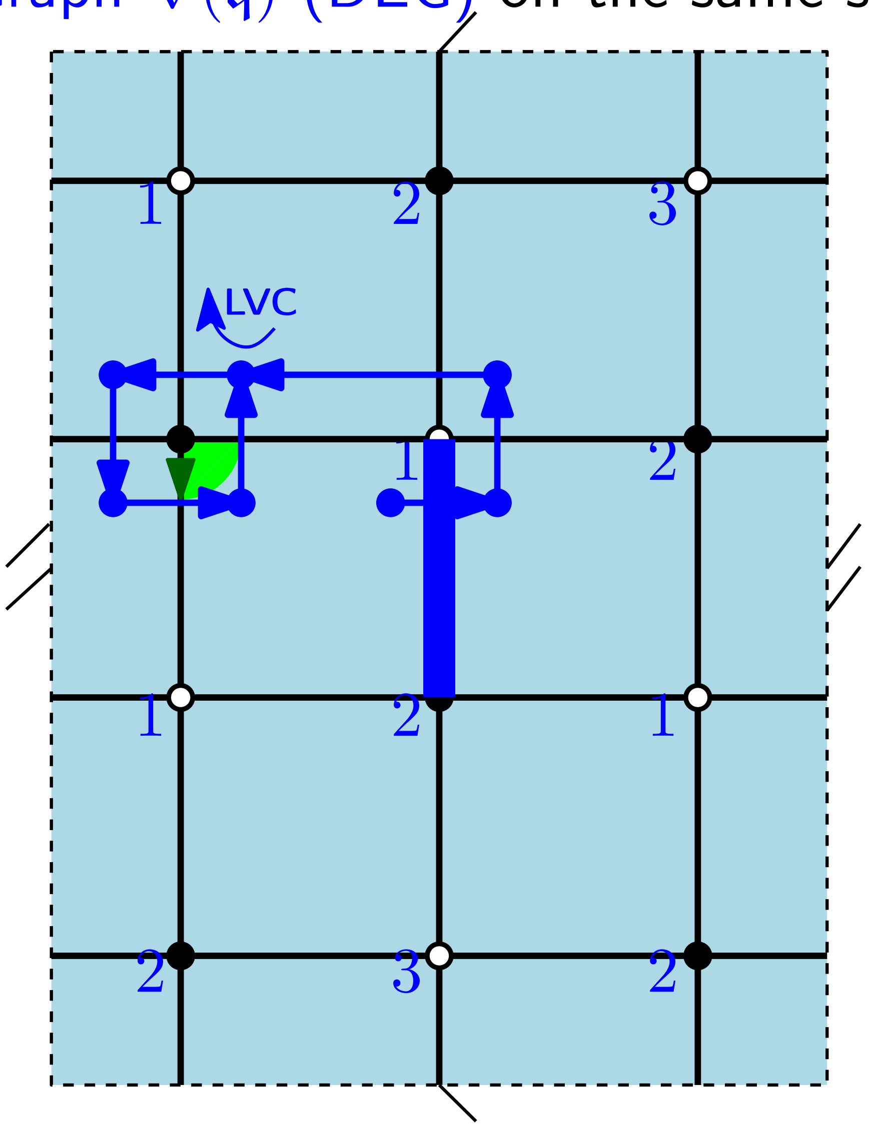

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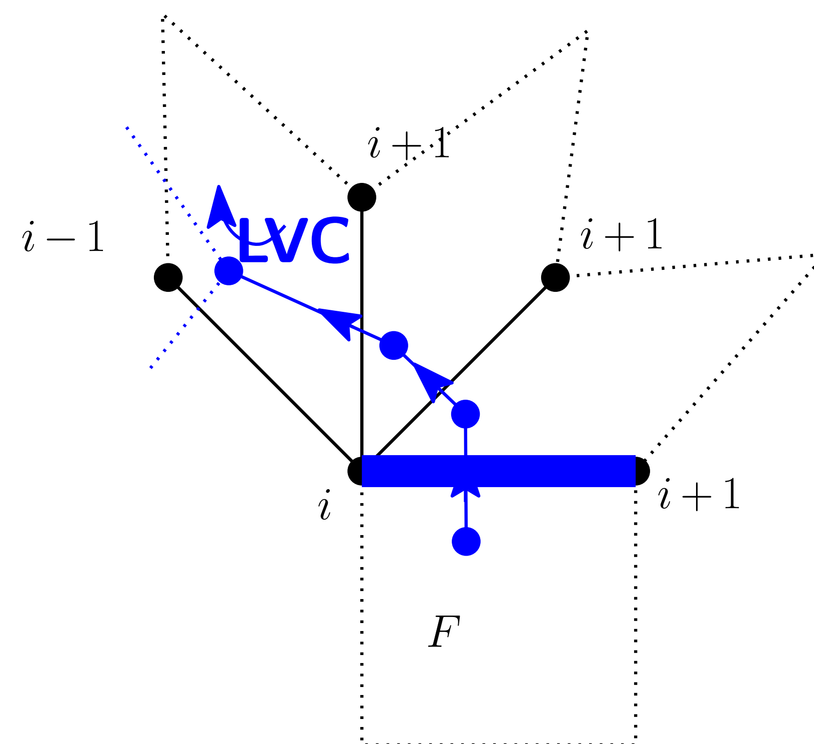

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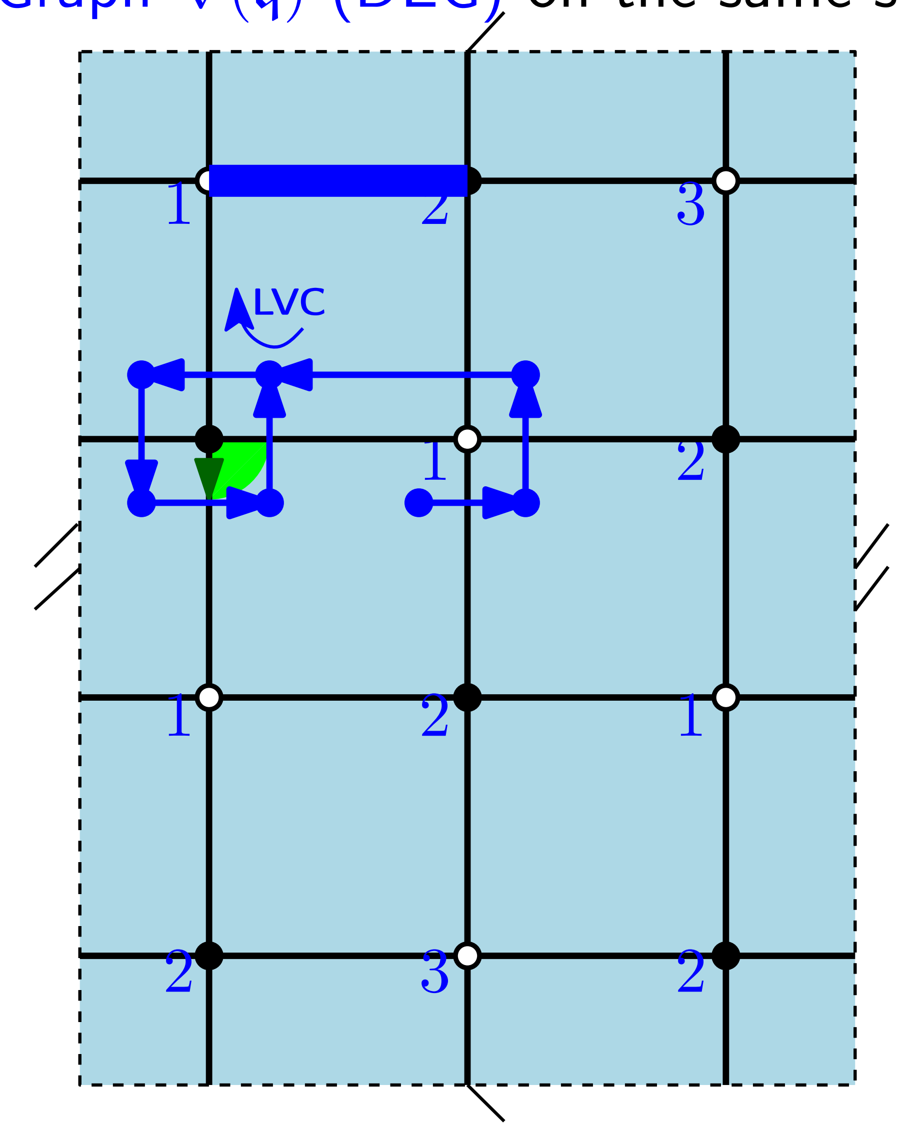

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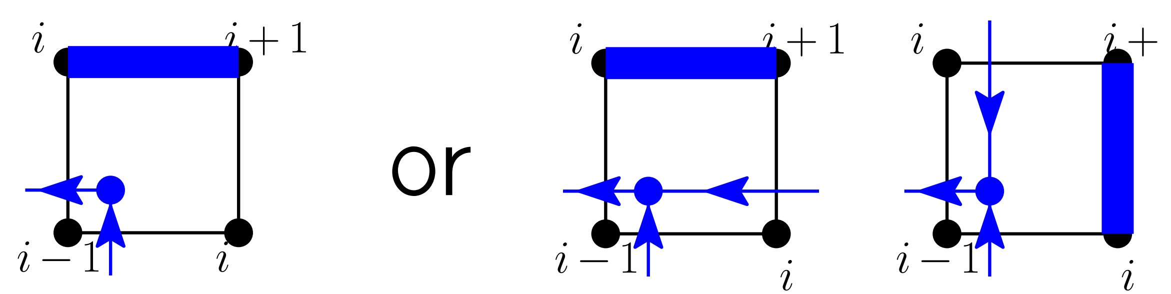

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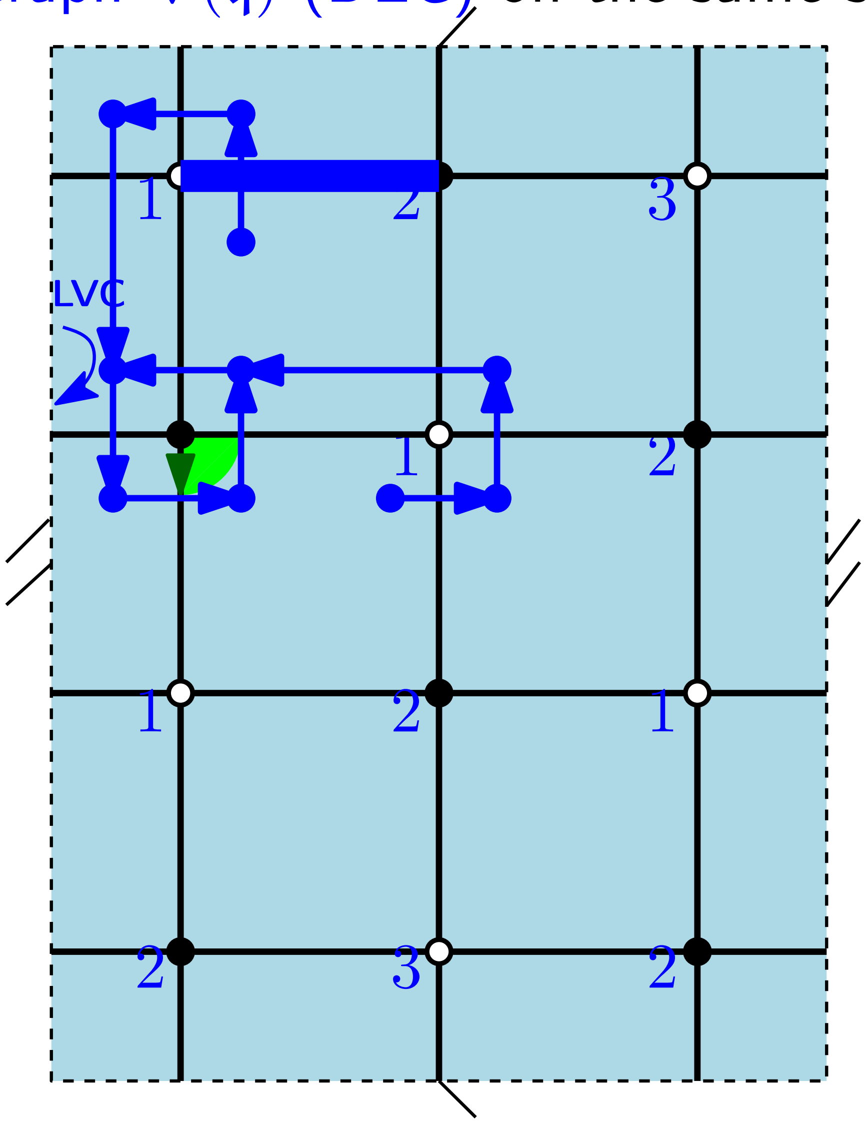

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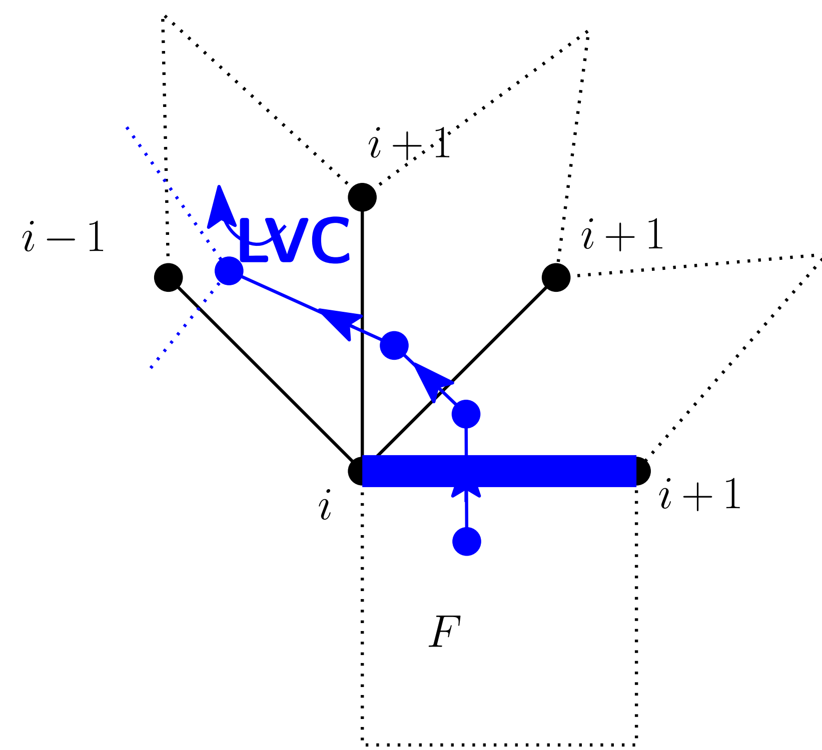

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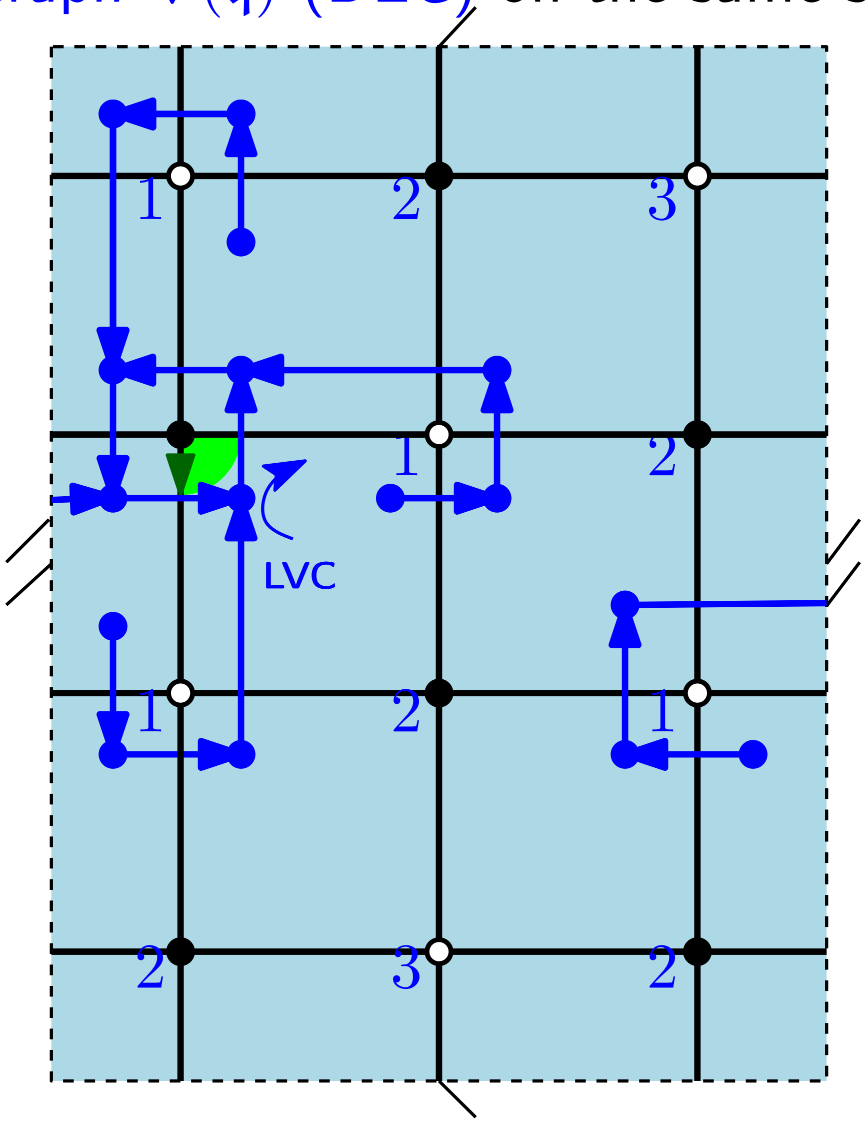

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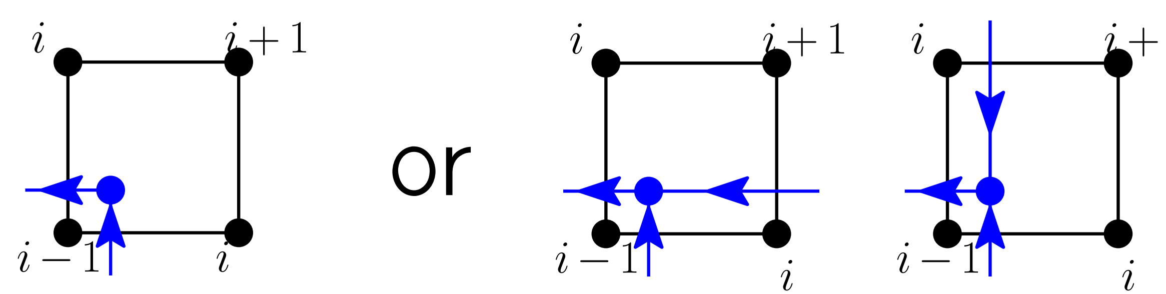

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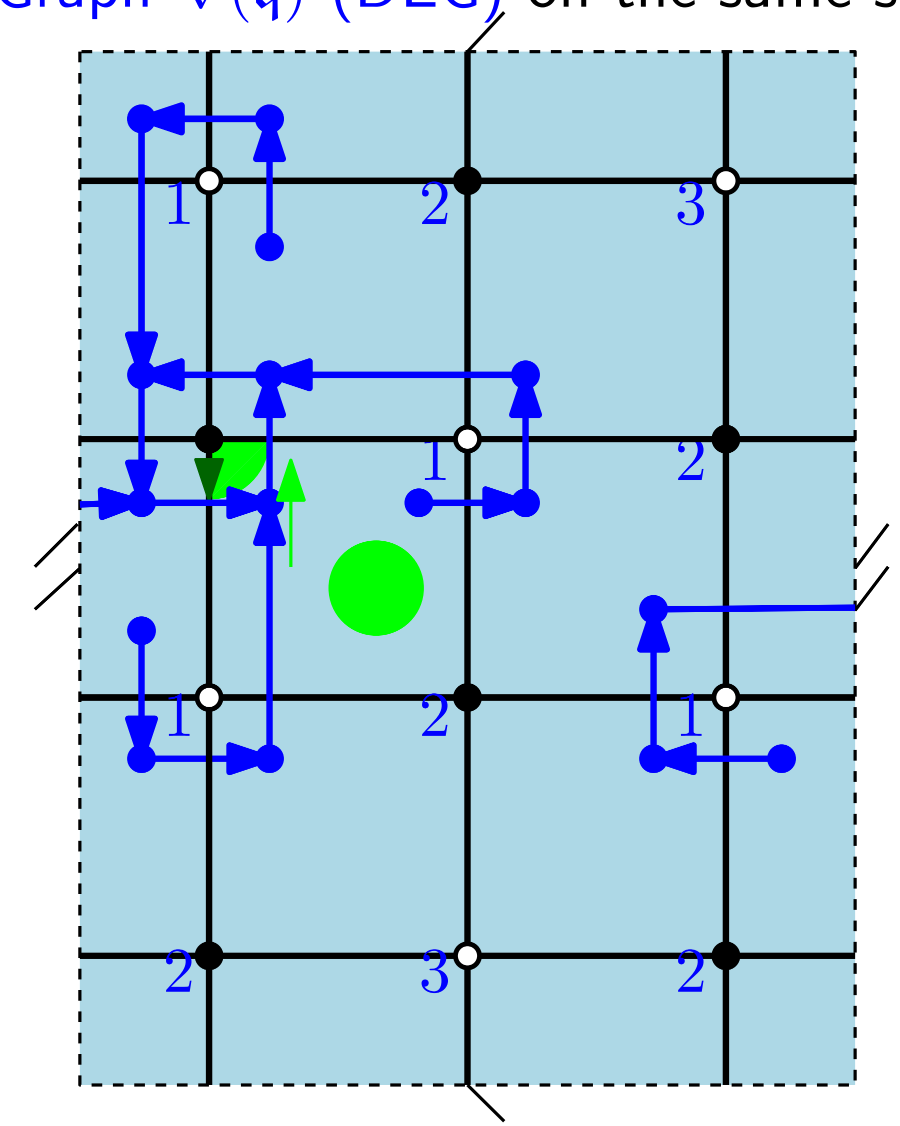

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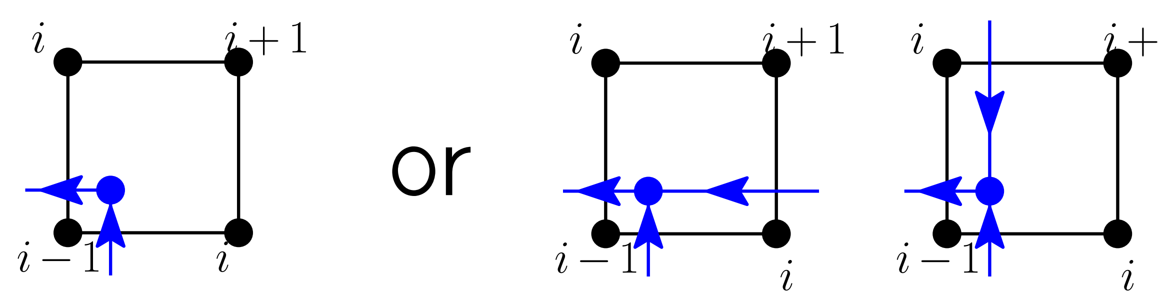

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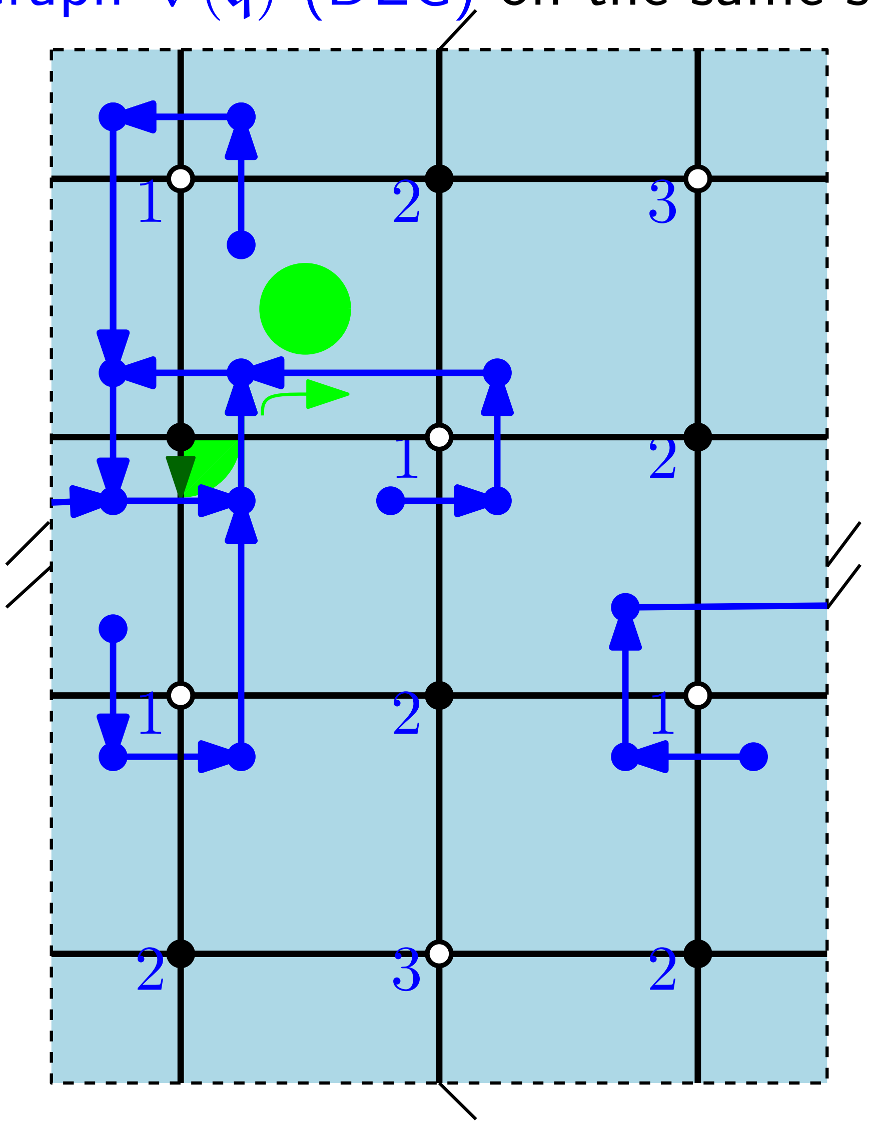

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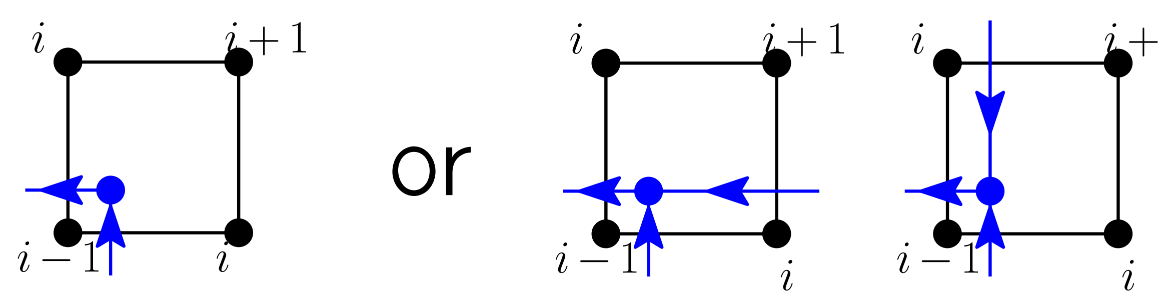

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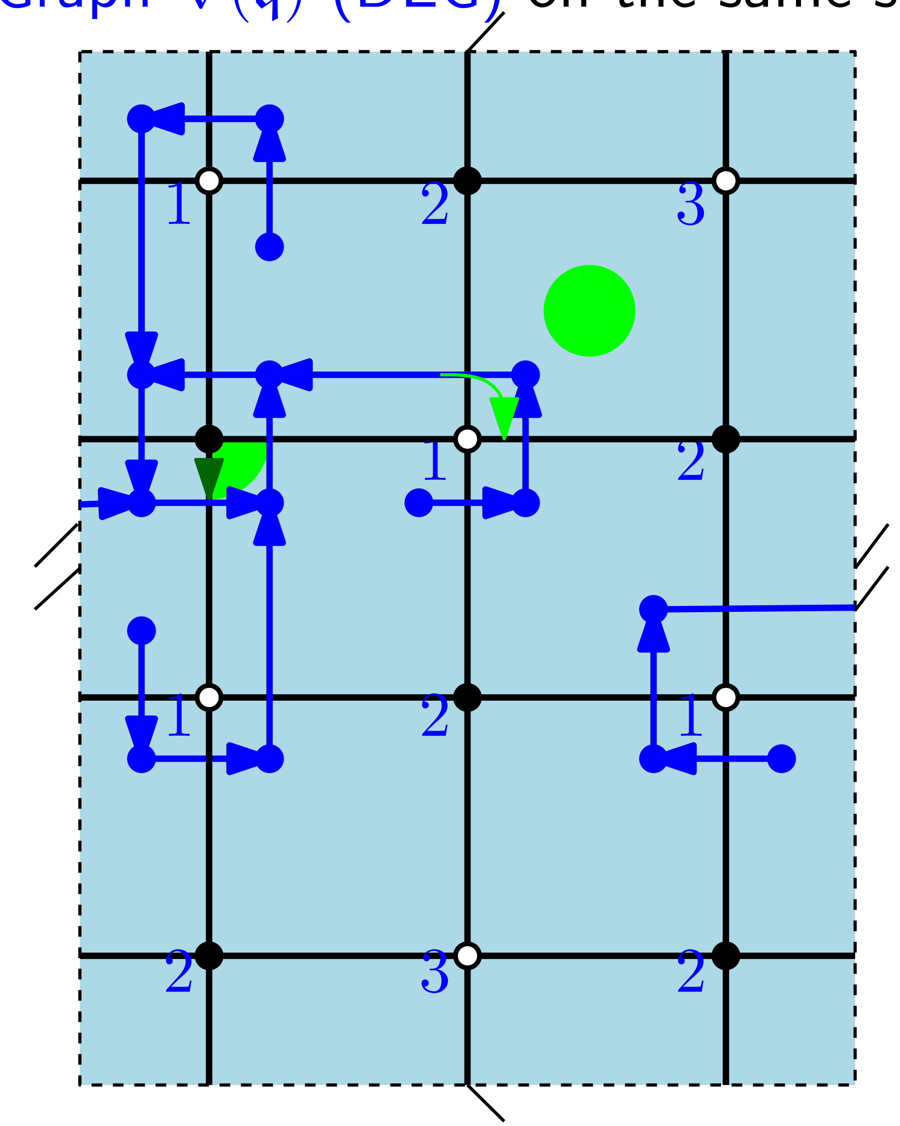

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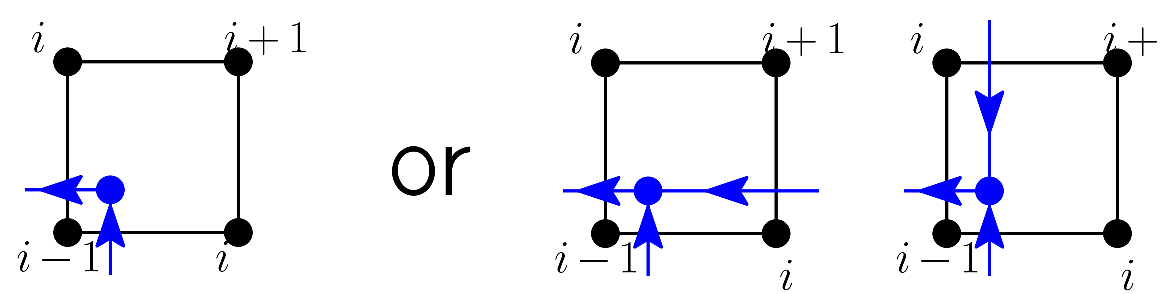

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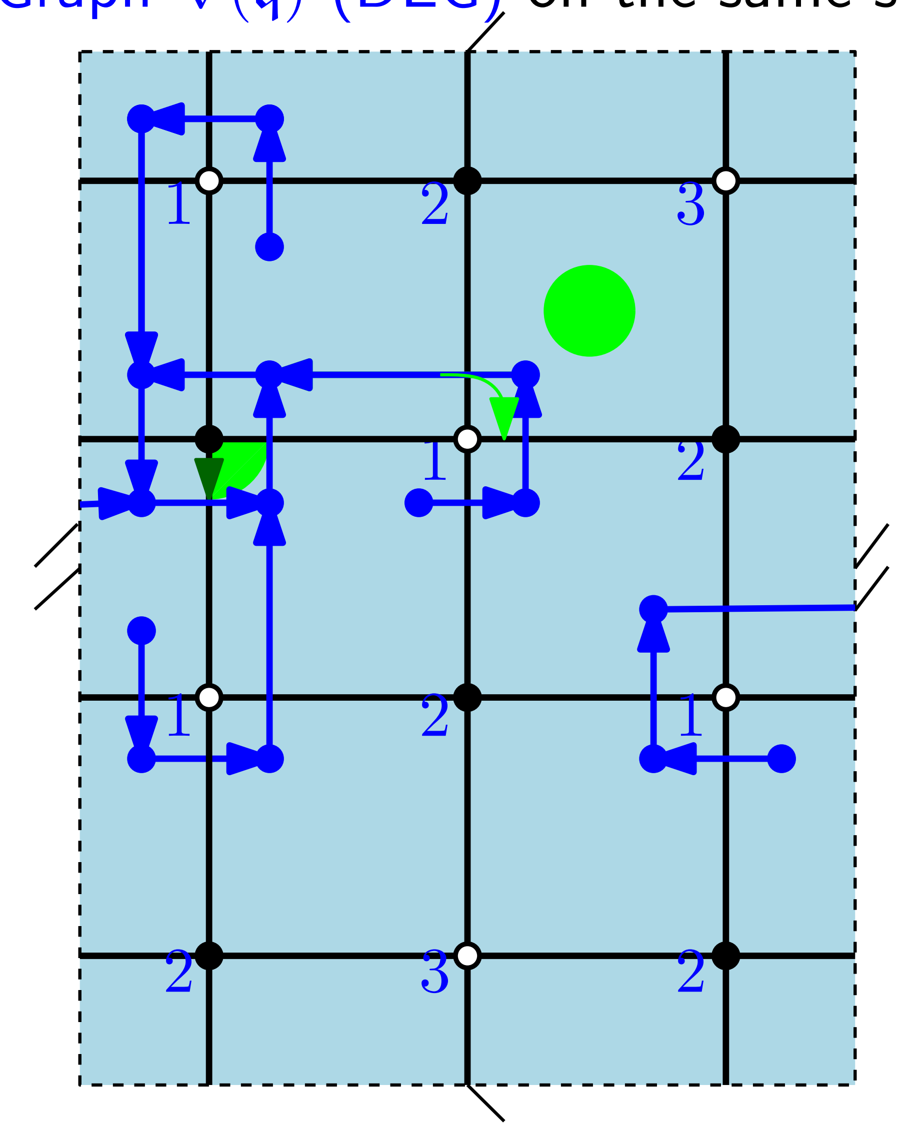

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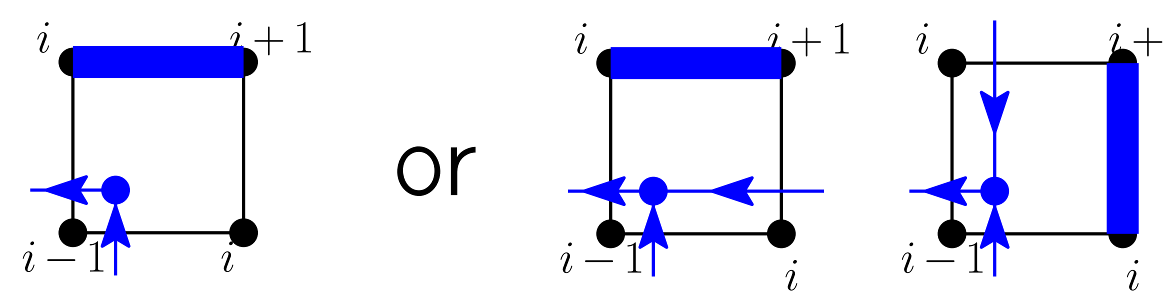

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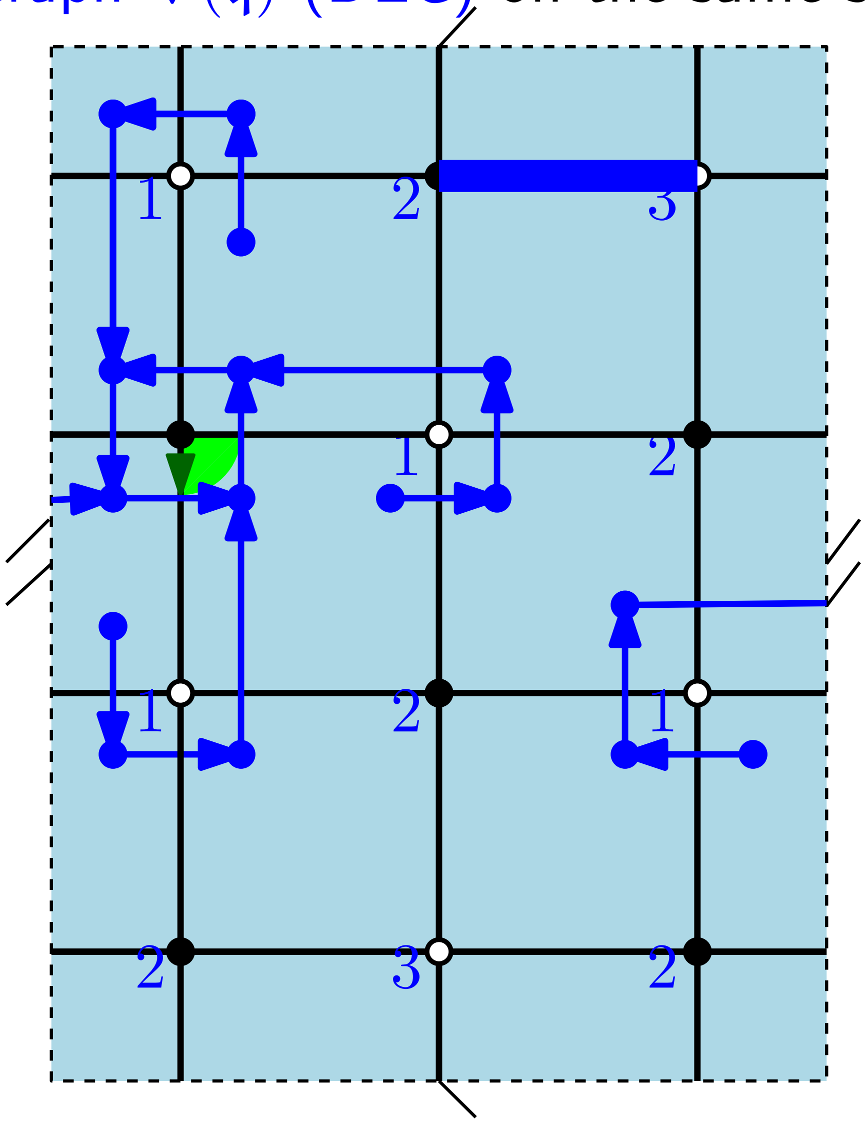

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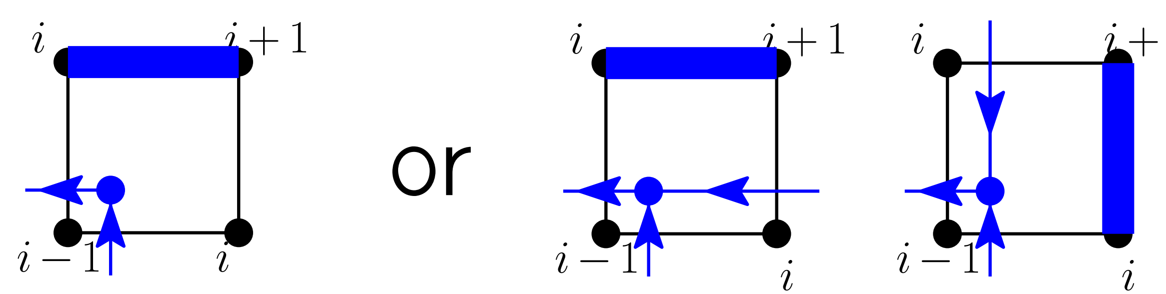

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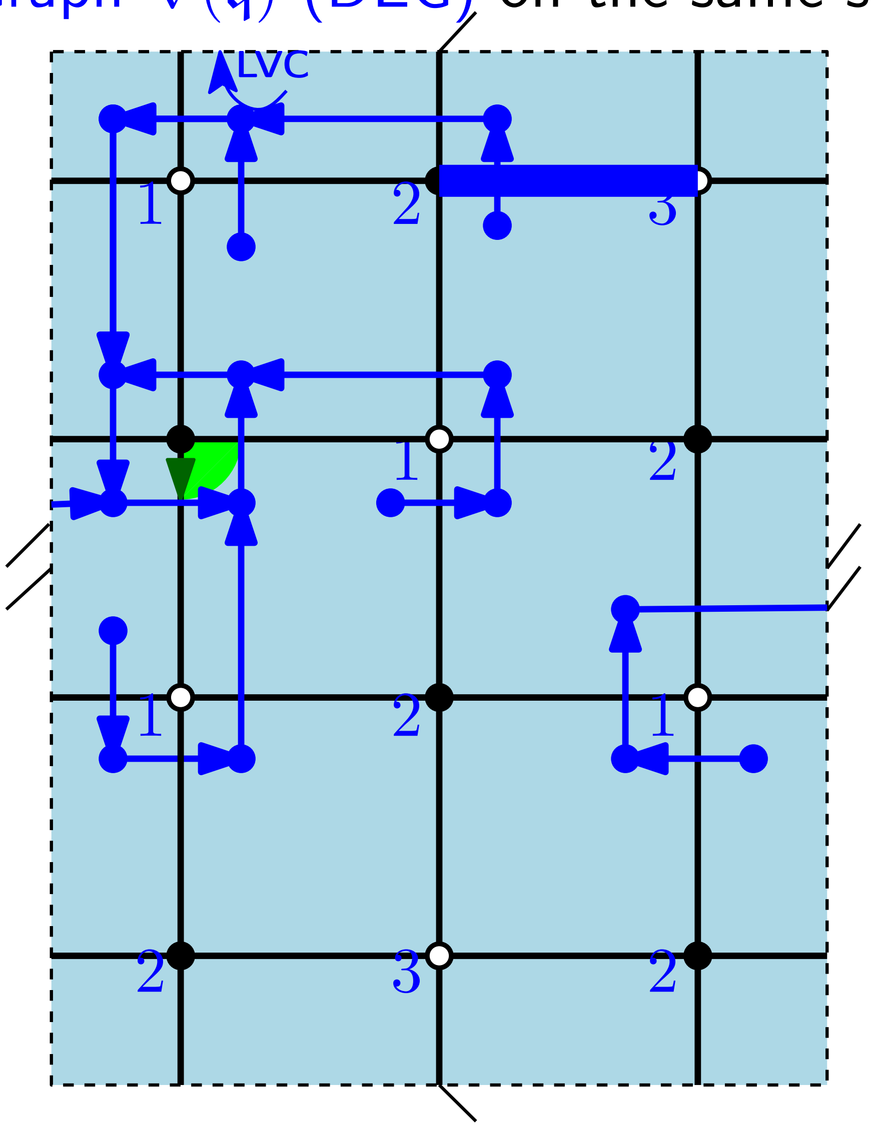

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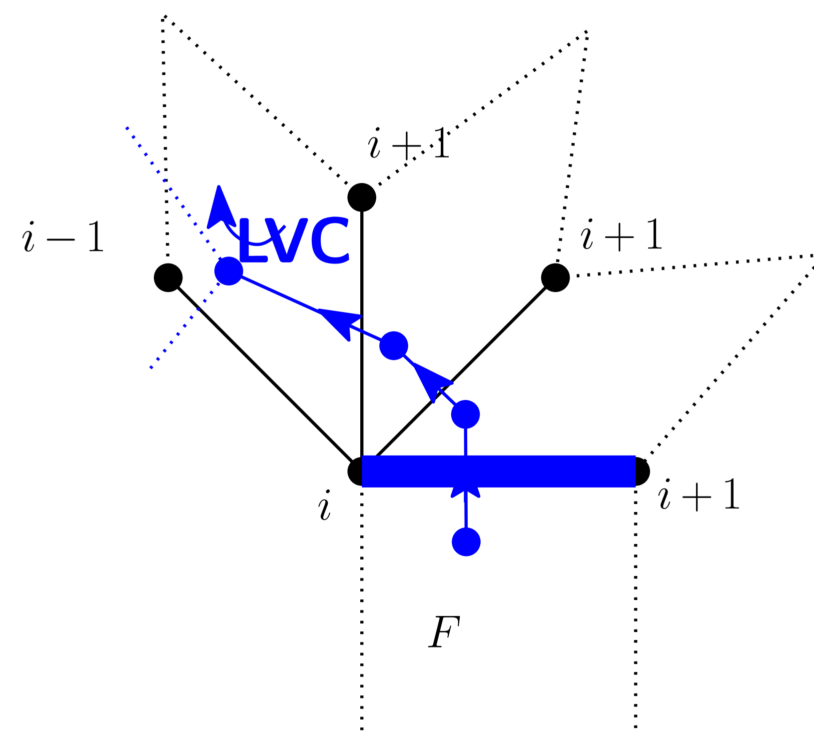

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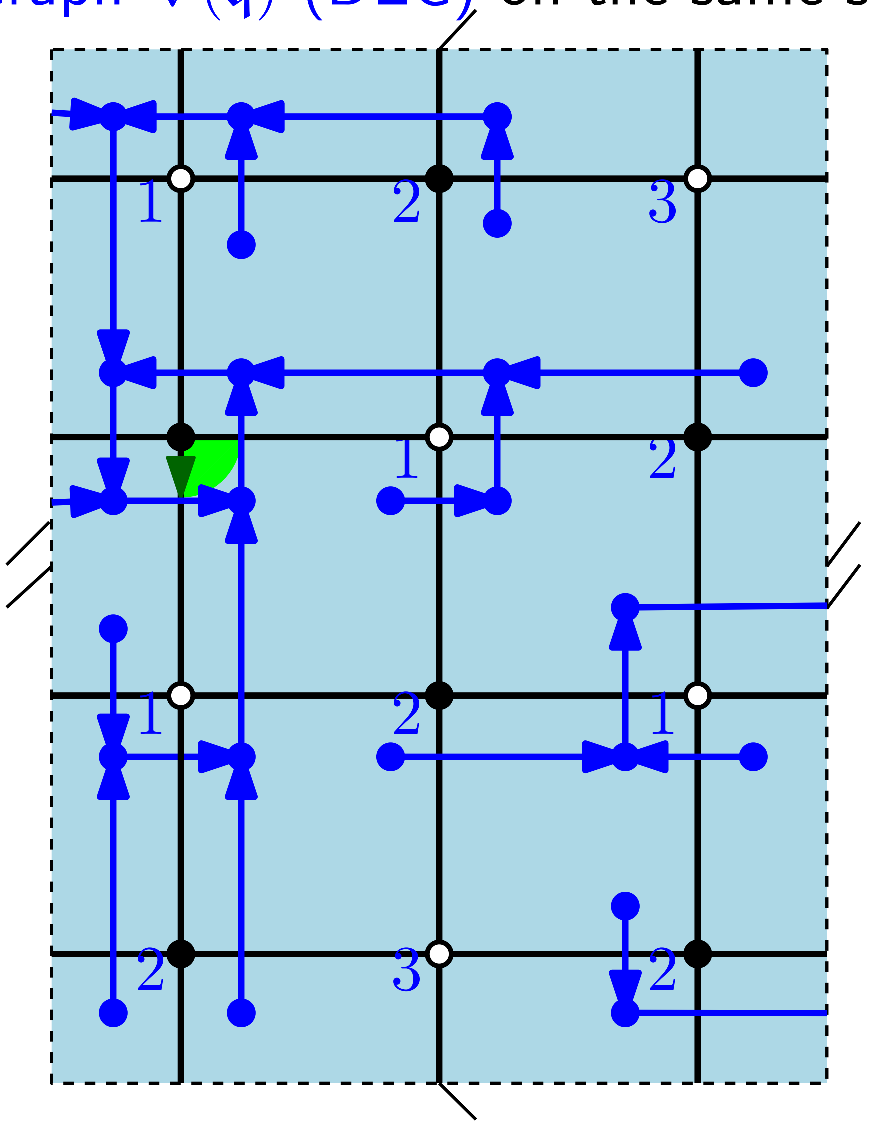

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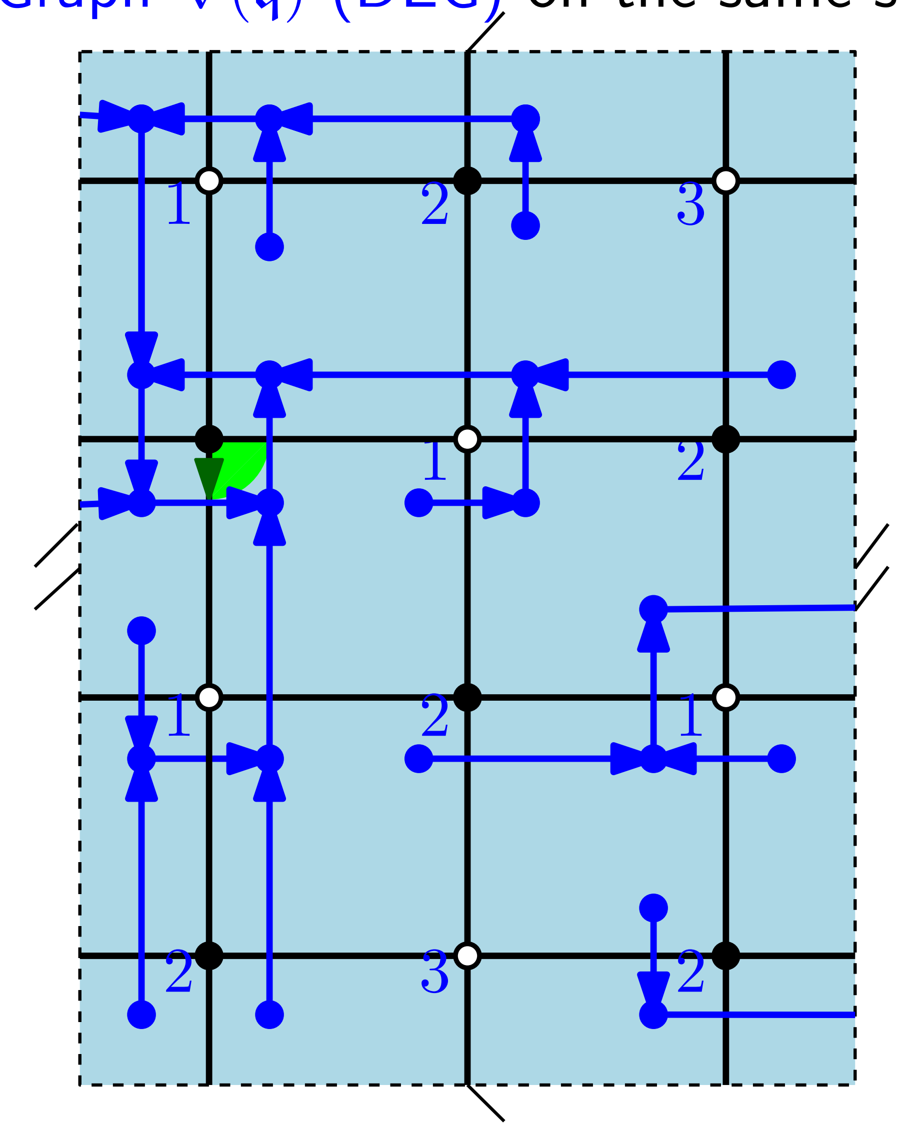
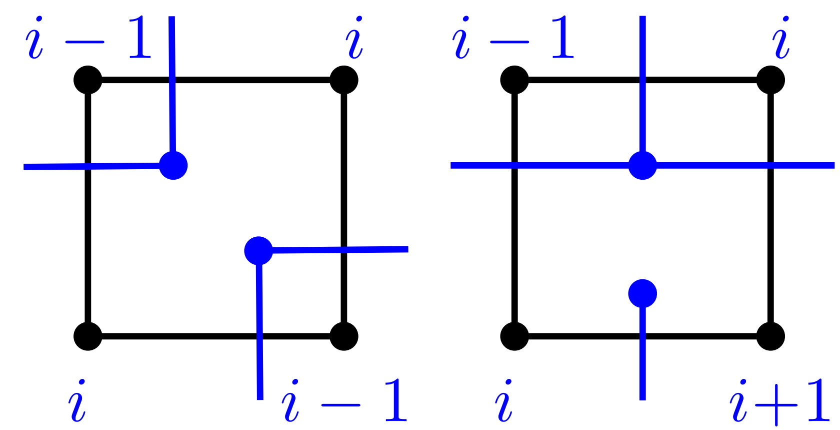


Proposition:
DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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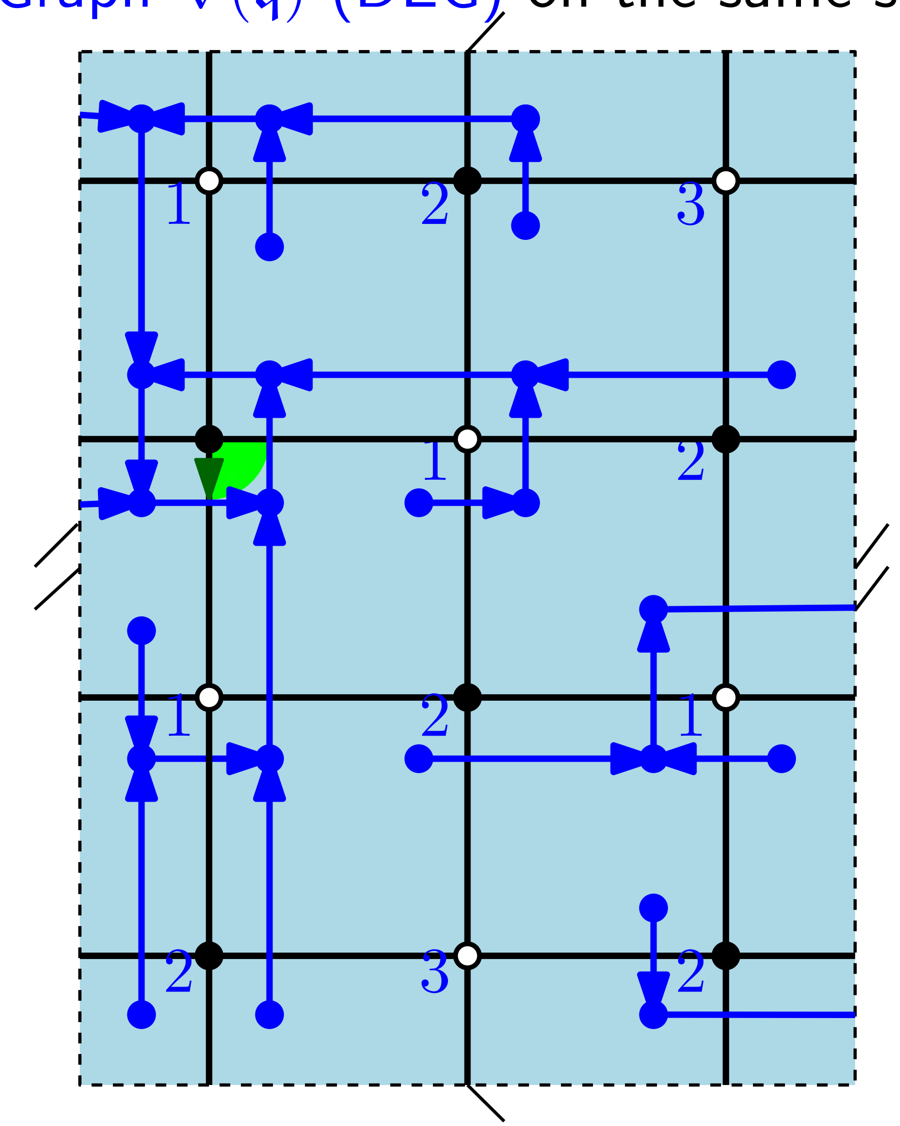
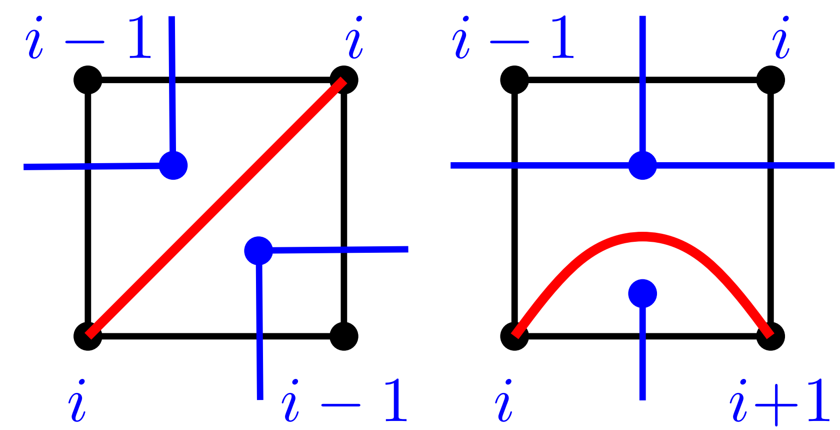


Proposition:

DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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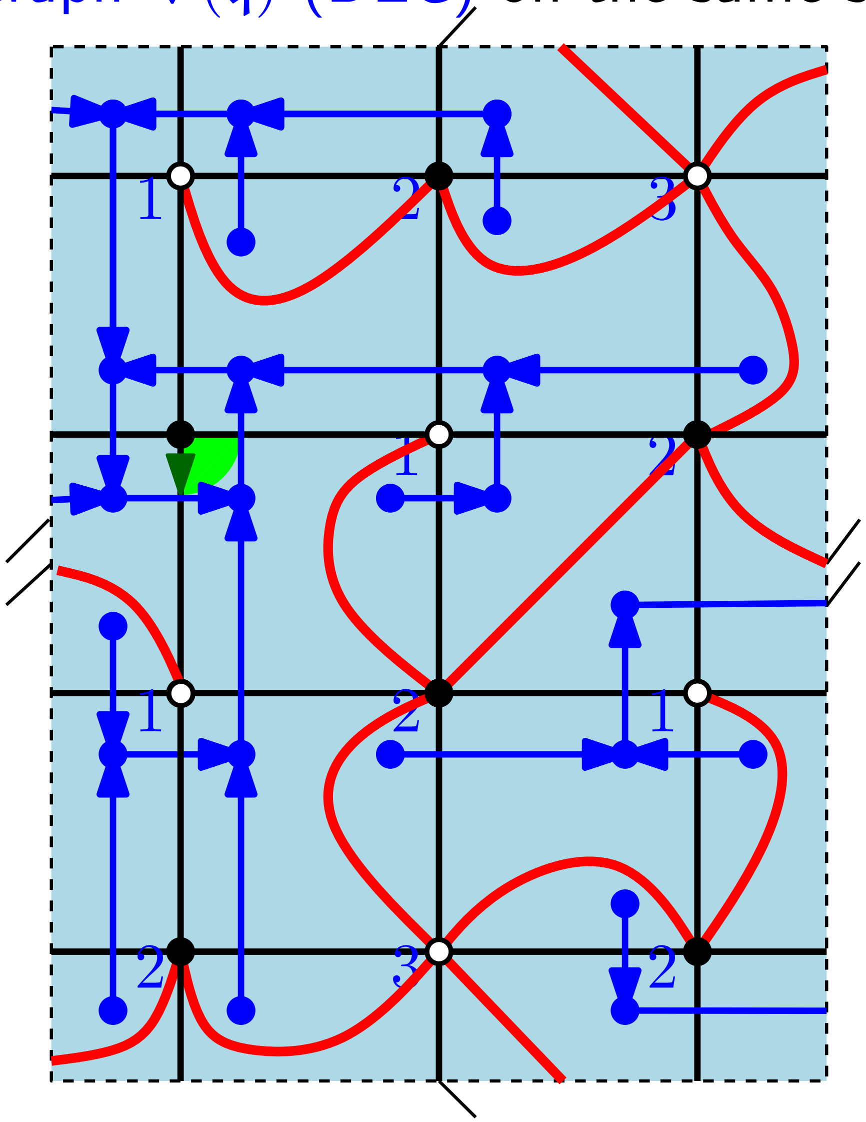

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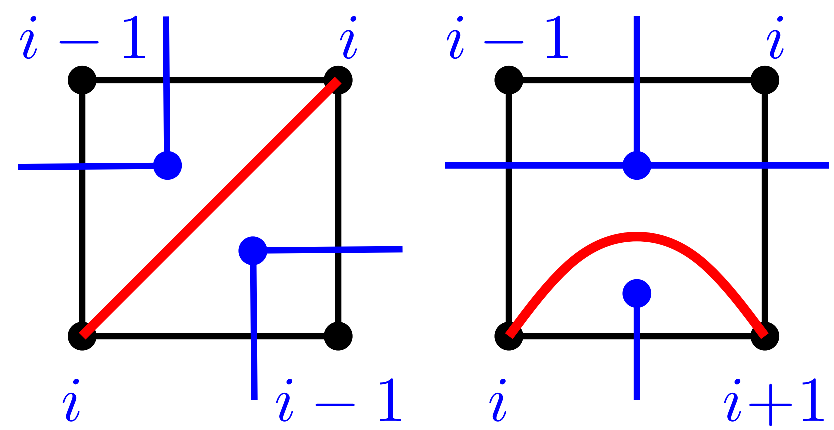

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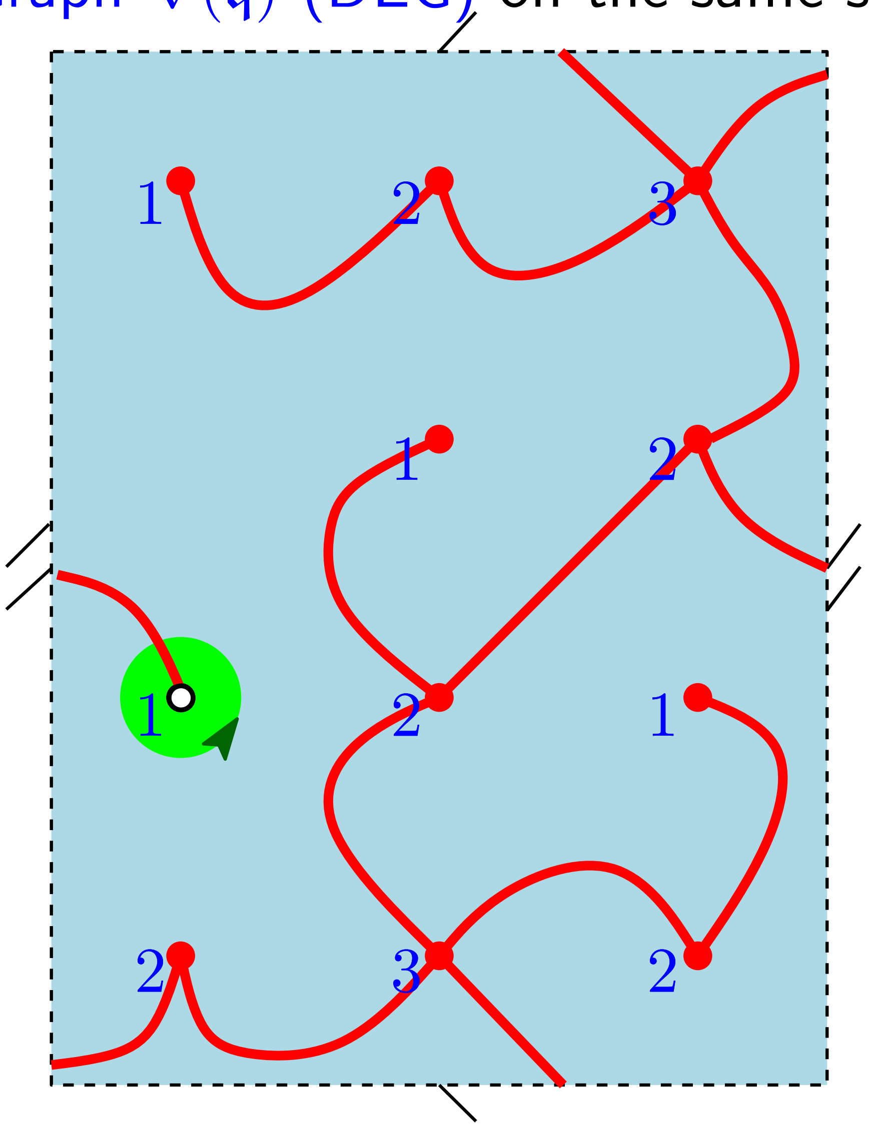

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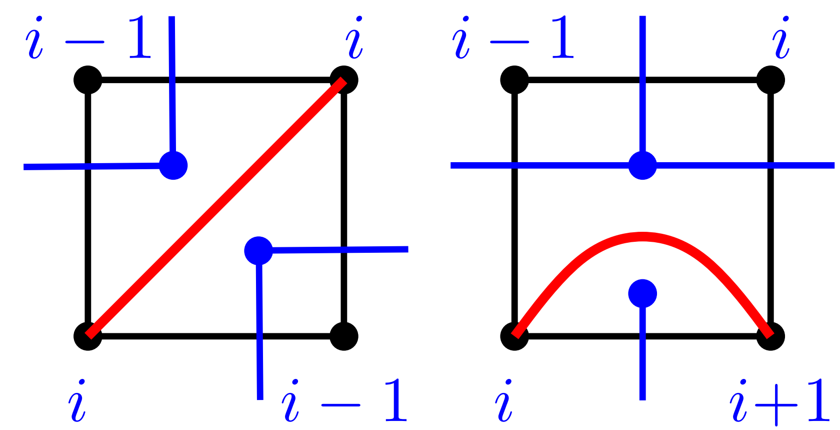

Corollary:
Red map $\phi(\mathfrak{q})$ is a one-face well-labeled rooted map with n edges, where n is the number of faces of $\mathfrak{q}$.

General case (III)

$\{$rooted, bipartite quadrangulations on $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)\}$

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Double rooting trick and Hall's marriage theorem!

Applications - enumeration

Theorem [Bender, Canfield 1986]
Let
$$Q_{\mathbb{S}}(t) := \sum_{n\geq 0} \vec{q}_{\mathbb{S},\bullet} t^n = \sum_{n\geq 0} (n+2-2g)\vec{q}_{\mathbb{S}}(n) t^n$$
be the generating function of rooted maps of type g pointed at a vertex or a face, by the number of edges. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2, \qquad U = tT^2(1 + U + U^2)$.
Then $Q_{\mathbb{S}}(t)$ is a rational function in U.

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Corollary [Bender, Canfield 1986]
When $\chi(\mathbb{S}) = 2 - 2g$, then there exists a constant $c(\mathbb{S})$ such that the number $m_{\mathbb{S}}(n)$ of rooted maps with n edges on $\mathbb{S}$ satisfies:

$$m_{\mathbb{S}}(n) \sim c(\mathbb{S}) \cdot n^{5(g-1)/2} 12^n.$$

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Remark
Our main theorem allows us to recover Bender and Canfield results (that was already recovered using combinatorial methods in the orientable case [Chapuy, Marcus, Schaeffer 2009]). In particular we can give some explicit (but very complicated) formula for the constant $c(\mathbb{S})$.

Applications - random maps

Let $(\mathcal{M}, v)$ be a map with distinguished vertex v. We define:

- radius of a map $\mathcal{M}$ centered at v by the quantity
$$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$
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Theorem [Chapuy, D. 2015]
Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on $\mathbb{S}$, let v_0 be a root vertex of q_n and let v_* be uniformly chosen vertex of q_n. Then, there exists a continuous, stochastic process $L^{\mathbb{S}} = (L_t^{\mathbb{S}}, 0 \le t \le 1)$ such that:

- $\left(\frac{9}{8n}\right)^{1/4} R(q_n, v_*) \to \sup L^{\mathbb{S}} - \inf L^{\mathbb{S}};$
- $\left(\frac{9}{8n}\right)^{1/4} d_{q_n}(v_0, v_*) \to \sup L^{\mathbb{S}};$
- $\frac{I_{(q_n, v_*)}\left((8n/9)^{1/4} \cdot\right)}{n+2-2h} \to \mathcal{I}^{\mathbb{S}},$

where $\mathcal{I}^{\mathbb{S}}$ is defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,
$$\langle \mathcal{I}^{\mathbb{S}}, g\rangle = \int_0^1 dt g(L_t^{\mathbb{S}} - \inf L^{\mathbb{S}}).$$

Further directions

• Generalization of the Bouttier-Di Francesco-Guitter bijection for non-orientable maps (bijection between bipartite $2p$-angulations, or, more generally bipartite maps with n faces of prescribed degrees and some kind of non-orientable mobiles?)

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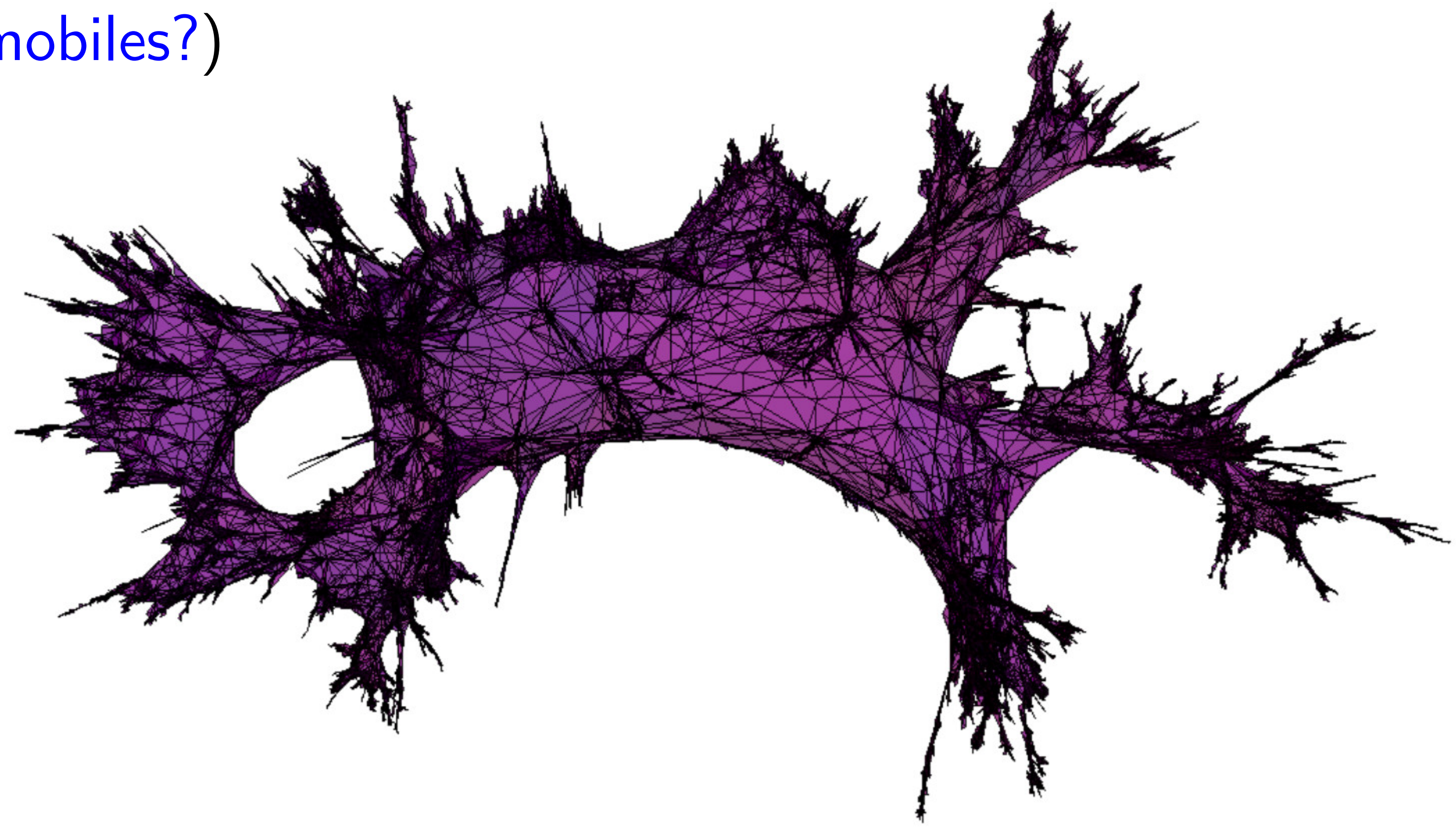

• Studying random maps on ANY surface in Gromov-Hausdorff topology (using our bijection and already established methods we (Bettinelli, Chapuy, D.) can prove a convergence of bipartite quadrangulations up to extraction of SUBSEQUENCE - what about full convergance)?).

THANK

YOU!