Coxeter elements in well-generated reflection groups

Vivien Ripoll

(Universität Wien)

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joint work with
Vic Reiner (Minneapolis) and **Christian Stump** (Berlin)

Introduction

- V vector space of dimension n, over $\mathbb{R}$ or $\mathbb{C}$
- W finite subgroup of $\mathrm{GL}(V)$ generated by reflections
  ($r \in \mathrm{GL}(V)$ of finite order and fixing pointwise a hyperplane)

Finite real reflection groups correspond to finite Coxeter groups.

Complex reflection groups are more... complex.
We consider only those that are well-generated, i.e., can be generated by n reflections.

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- $R := \{$all reflections of $W\}$
- $\ell_R : W \to \mathbb{N}$ reflection length (or absolute length)

Coxeter elements are special elements c of W such that

- $\ell_R(c) = n$
- c is product of a "nice" generating set of n reflections.

Motivations:

- Coxeter-Catalan combinatorics
- understanding presentations for complex reflection groups

Questions:

- What is exactly a Coxeter element?
- How are different Coxeter elements related?
- Which role do these Coxeter elements play in the geometry and combinatorics of reflection groups?

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Outline

	Classical definition	Extended definition
1. W real reflection group $\rightarrow$ using Coxeter generators	**1.a.**	**1.b.**
2. W complex reflection group $\rightarrow$ using eigenvalues	**2.a.**	**2.b.**

3. Reflection automorphisms and main results

Outline

1. Coxeter elements in real reflection groups — *via* Coxeter systems
   - Classical definition
   - Extended definition
2. Coxeter elements in well-generated complex reflection groups — *via* eigenvalues
   - Classical definition
   - Extended definition
3. Reflection automorphisms and main results

Real reflection group and Coxeter system

Let W be a **real** reflection group.

Choose a chamber $\mathcal{C}$ of the hyperplane arrangement of W. Define

$$S := \{\text{reflections through the walls of } \mathcal{C}\}.$$

Then:

- S generates W;
- (W, S) is a (finite) Coxeter system:

Coxeter system

The group W has a presentation of the form:

$$W = \left\langle S \;\middle|\; s^2 = 1 \ (\forall s \in S);\ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \right\rangle ,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2}$ for $s \neq t$.

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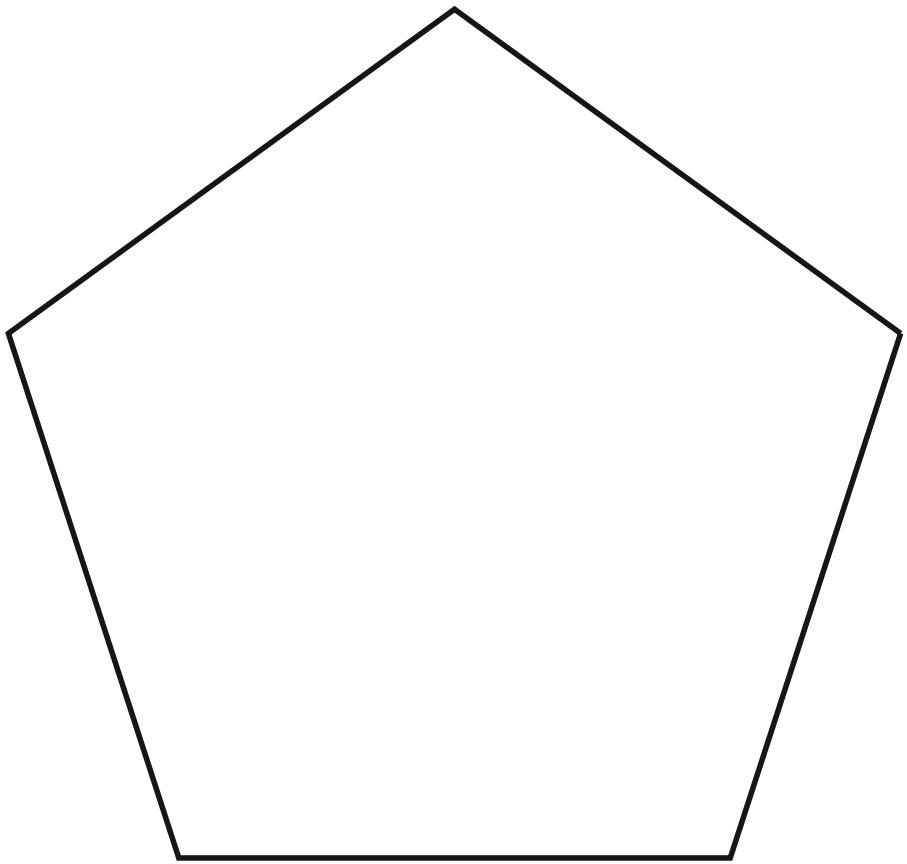

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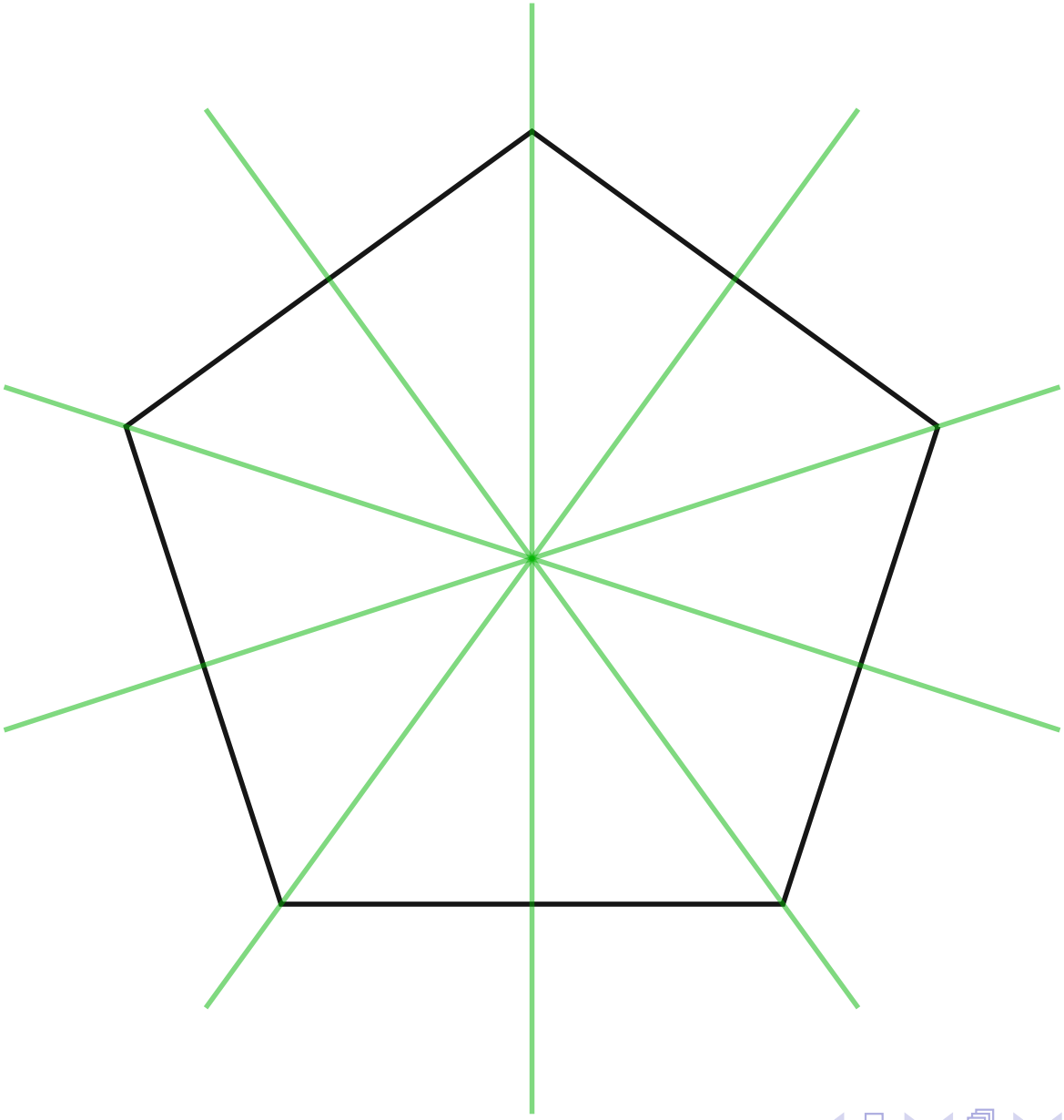

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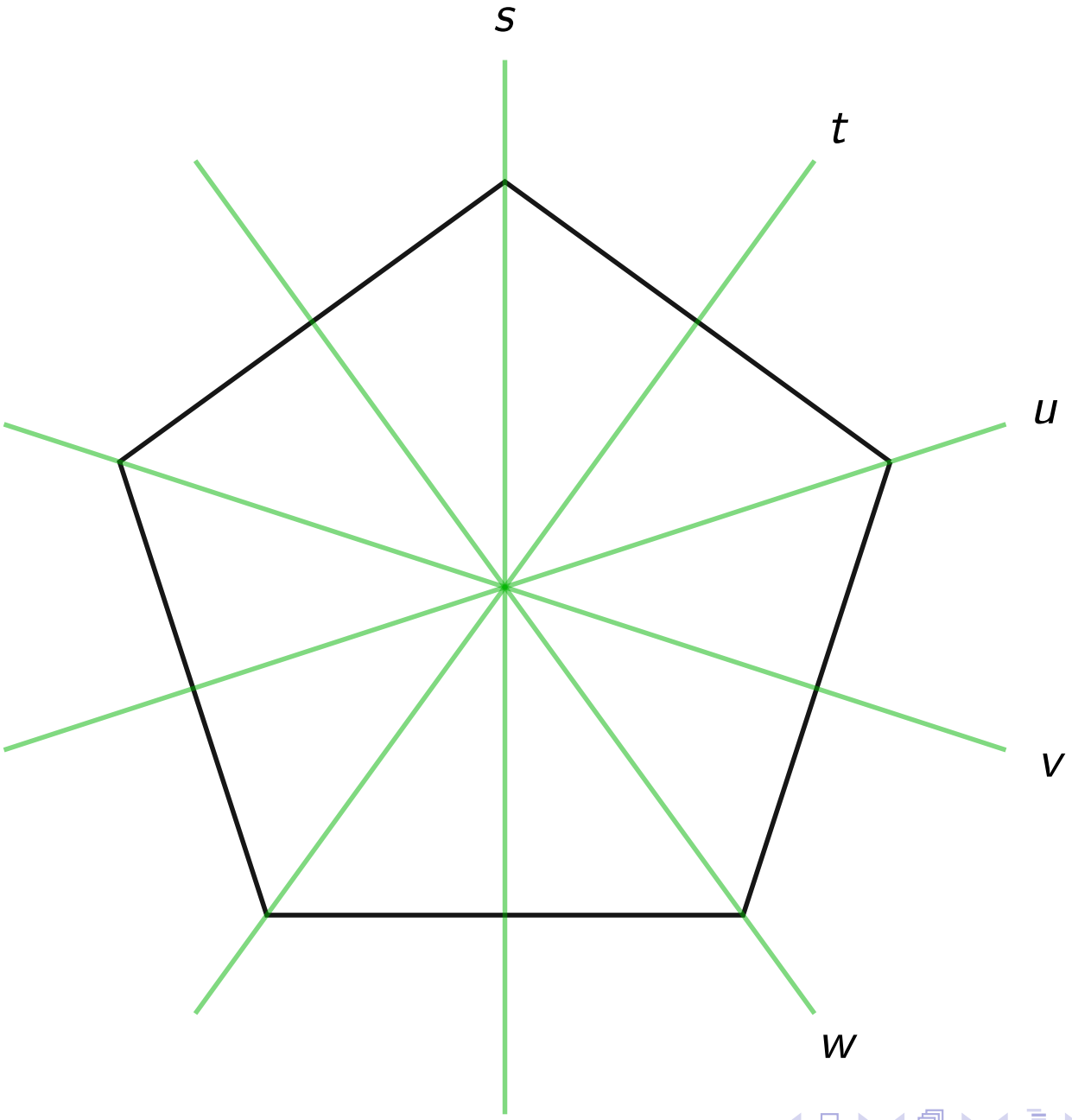

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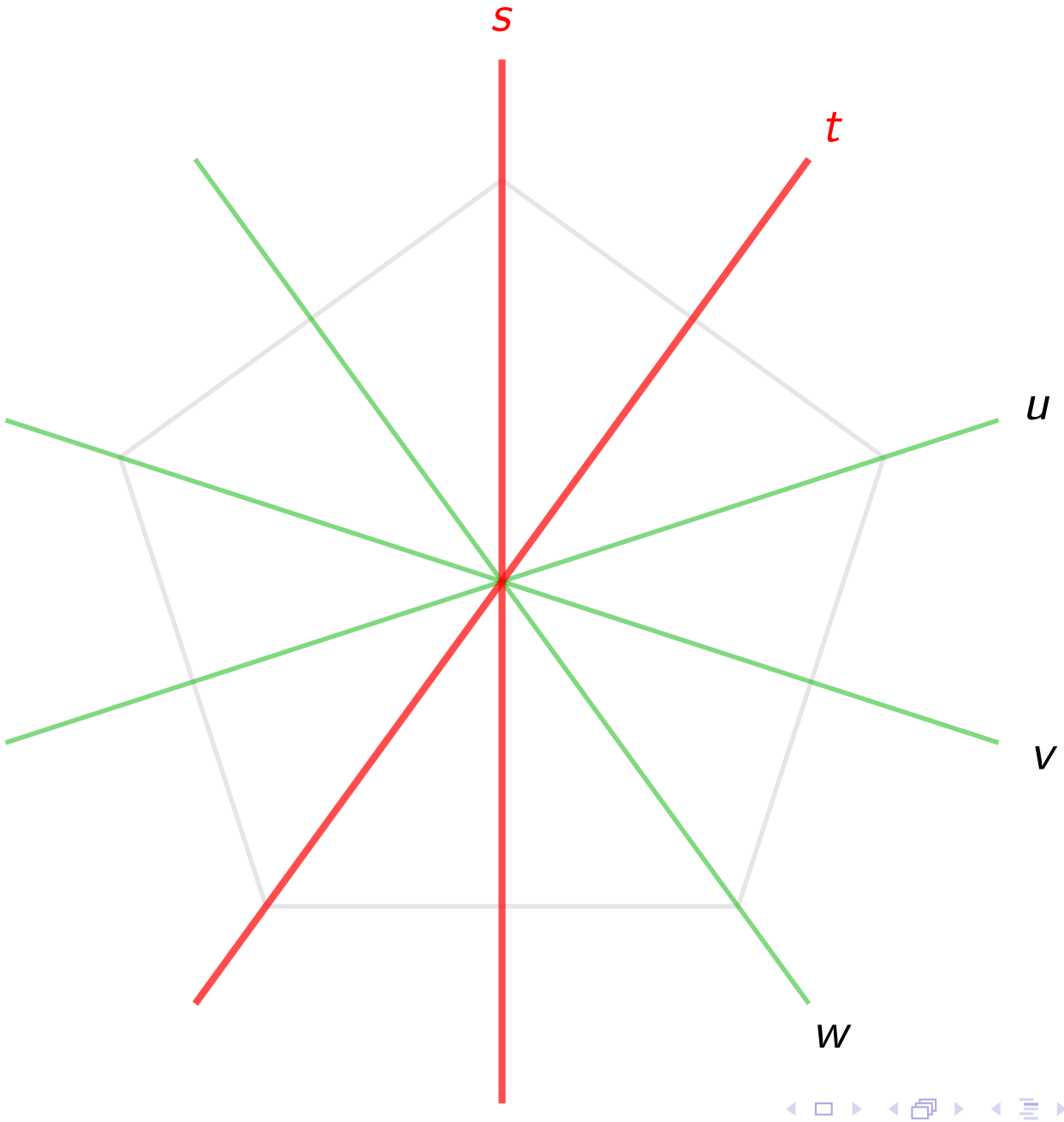

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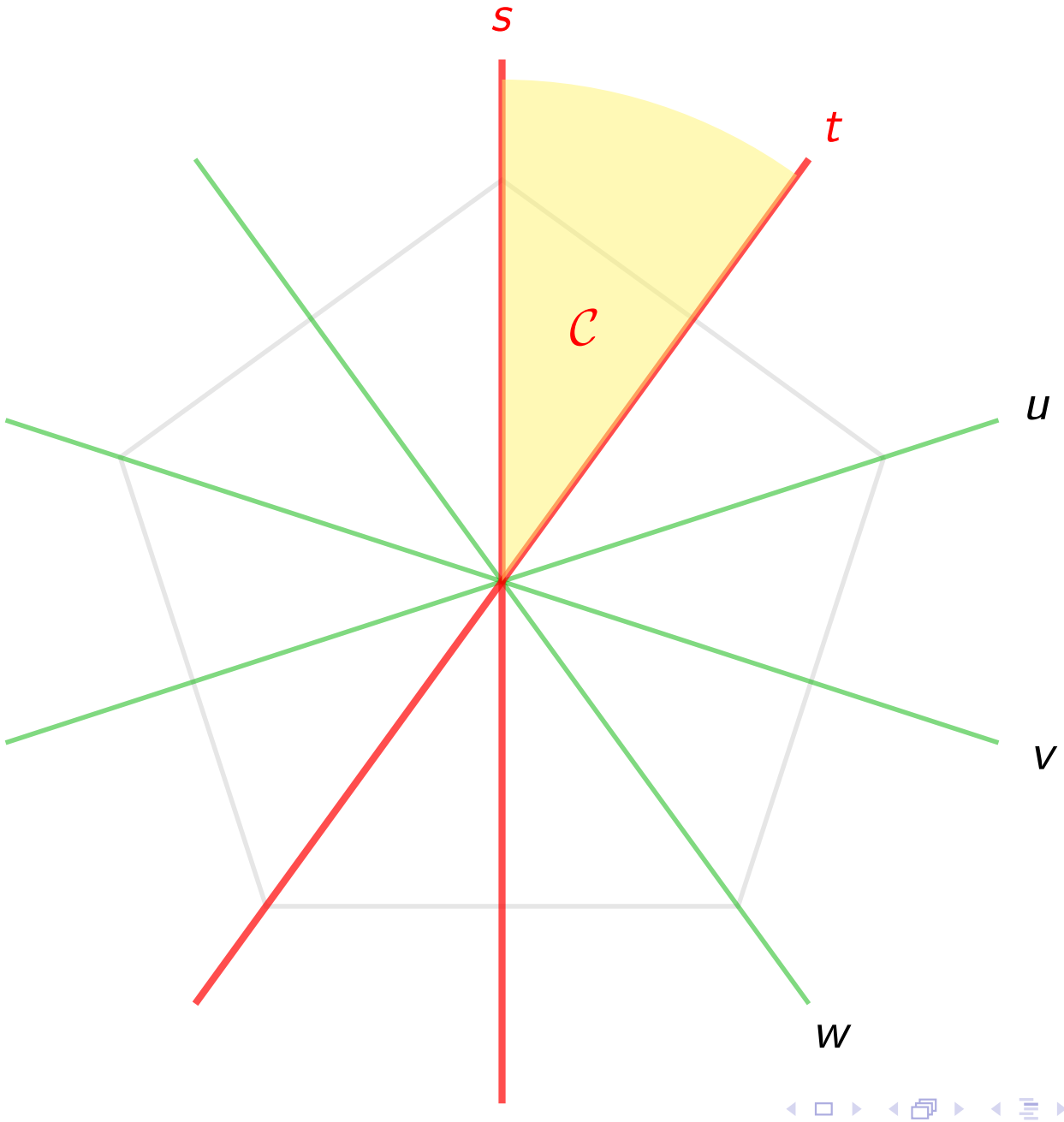

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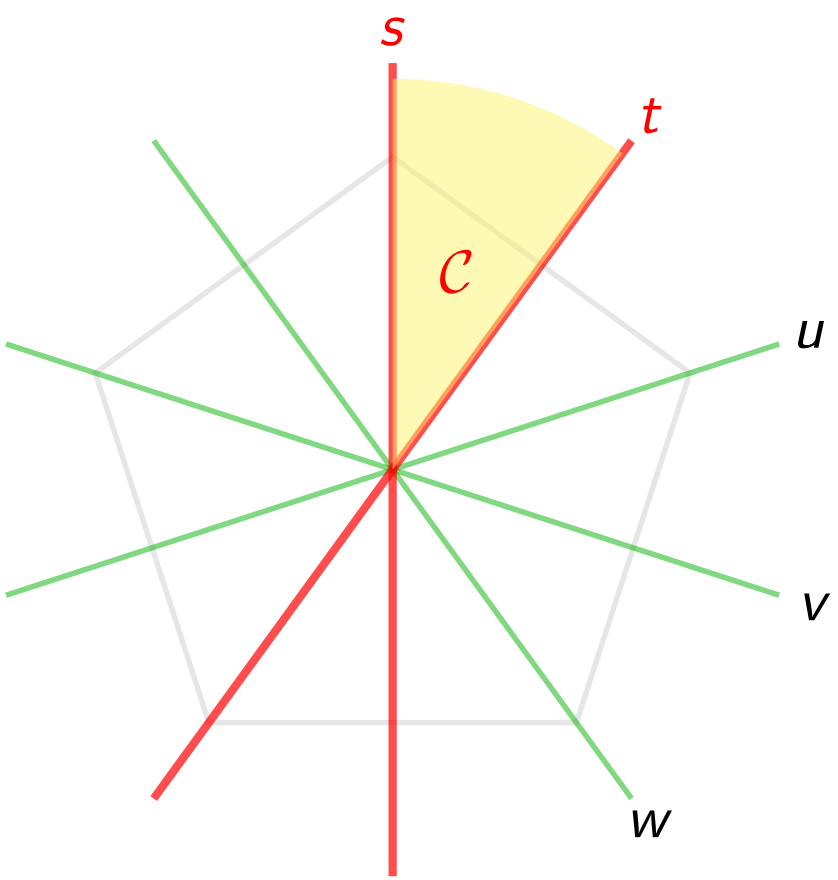

$$W = \langle s, t \mid s^2 = t^2 = 1, (st)^5 = 1 \rangle$$

Coxeter element = product of such generators, coming from the choice of a chamber:

- st, ts
- tu, ut
- uv, vu
- vw, wv
- ws, sw

$\rightsquigarrow$ only two distinct Coxeter elements: rotations of angle $\pm\frac{2\pi}{5}$.

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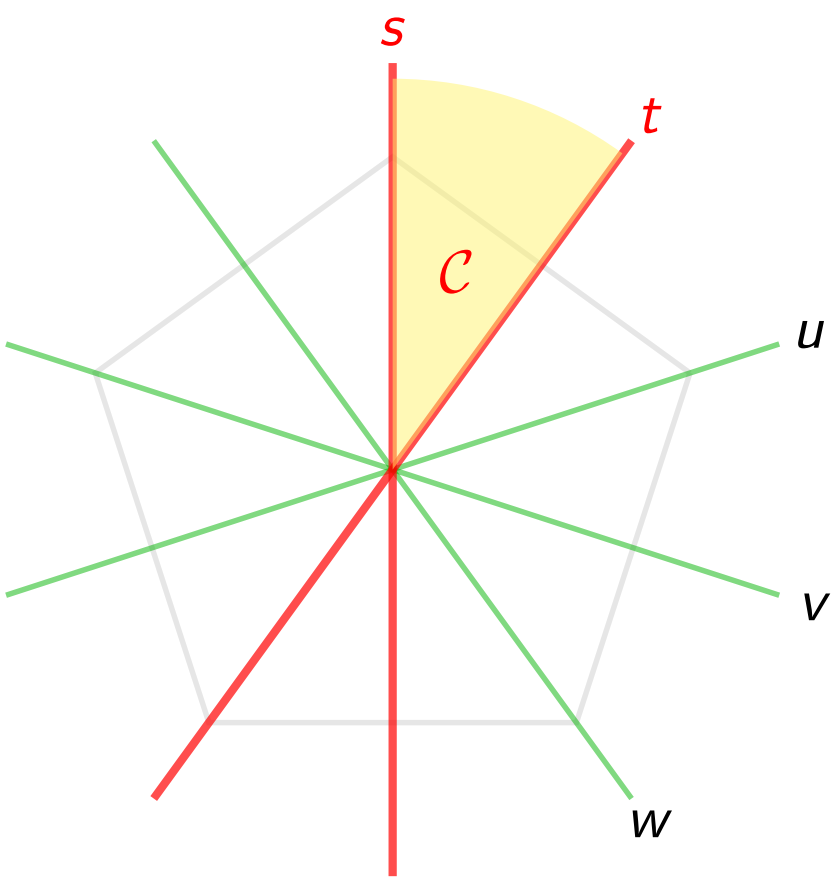

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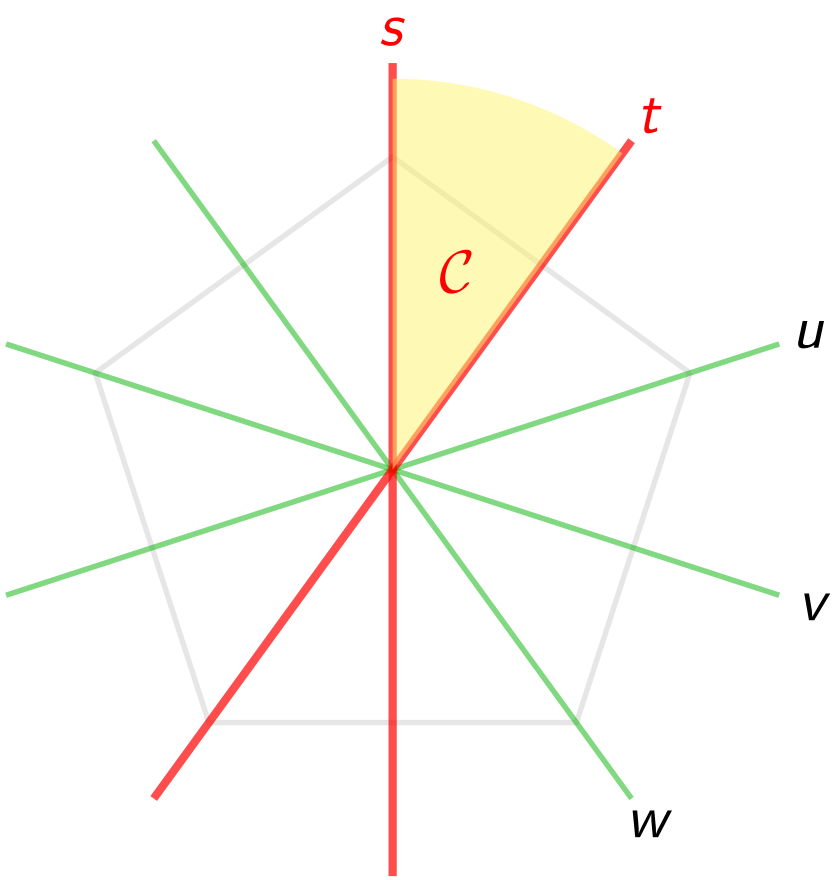

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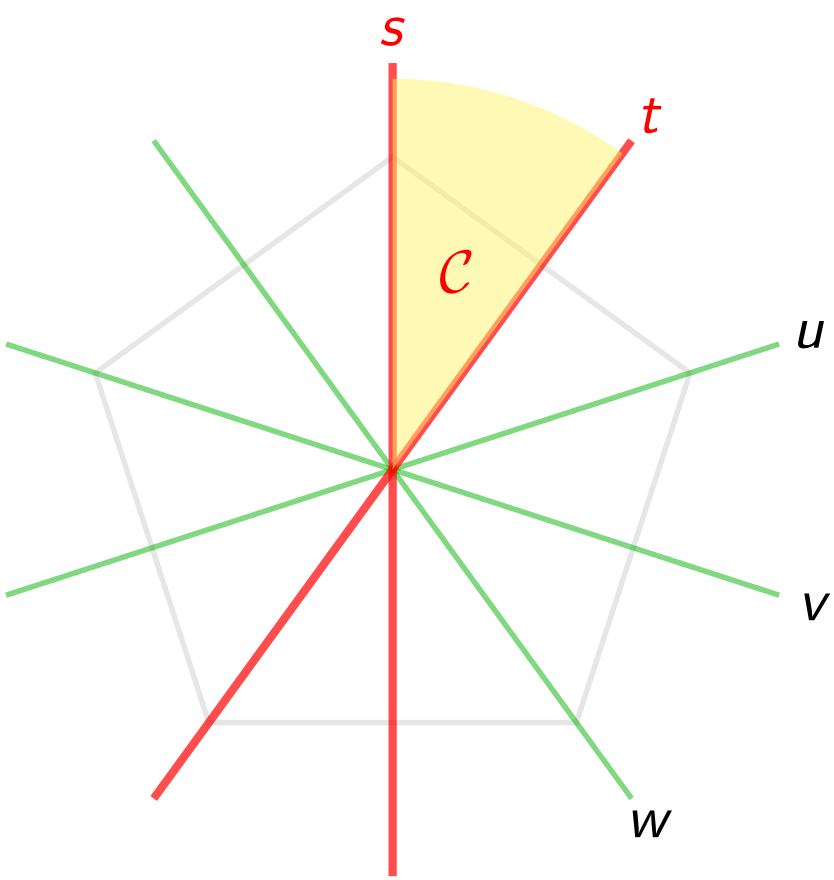

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Coxeter elements in a real reflection group

Definition (1.a.)

Let W be a finite real reflection group. A Coxeter element of W is a product (in any order) of all the reflections through the walls of a chamber of W.

	Classical definition	Extended definition
W real	Product of reflections through the walls of a chamber	...
W complex	...	...

Proposition

These Coxeter elements form a conjugacy class of W.

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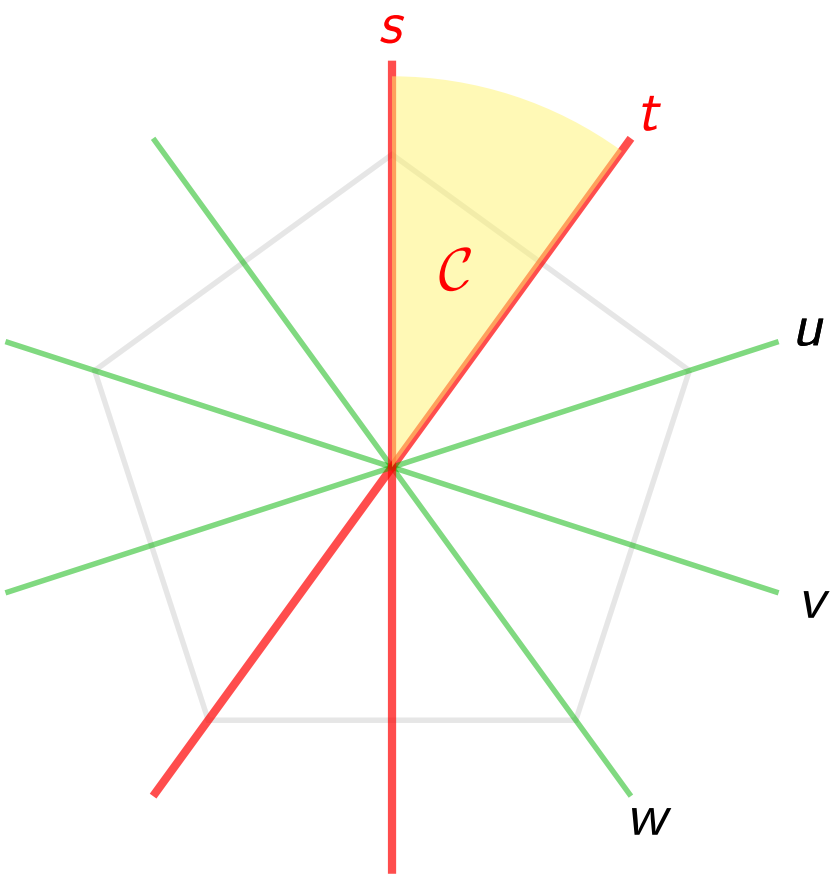
Dihedral group: unorthodox Coxeter elements

For a reflection group W, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...



Here take $S = \{s, u\}$.

$$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$$

$\rightsquigarrow$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\rightsquigarrow$ Define more general Coxeter elements = product of such generators

New Coxeter elements here:
su, us, tv, vt...
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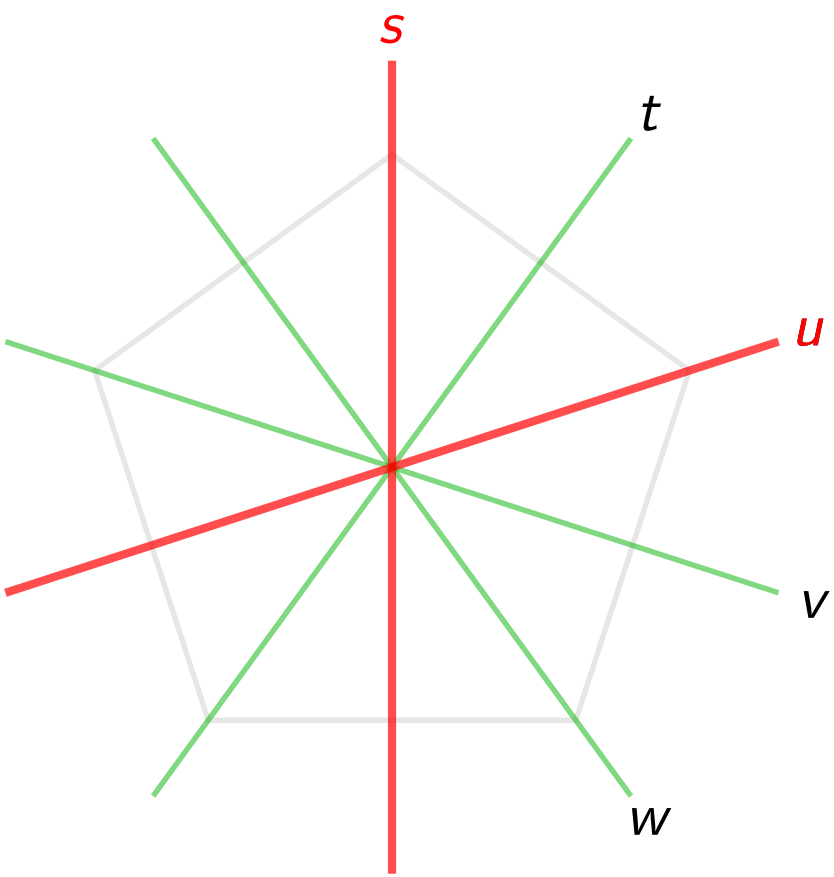

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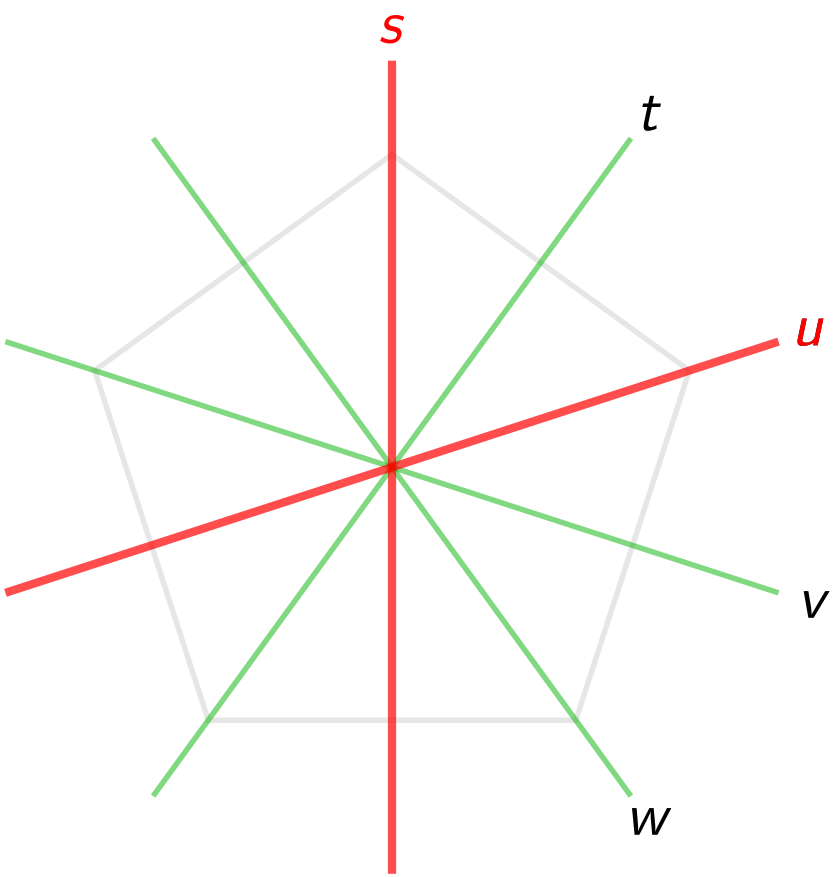

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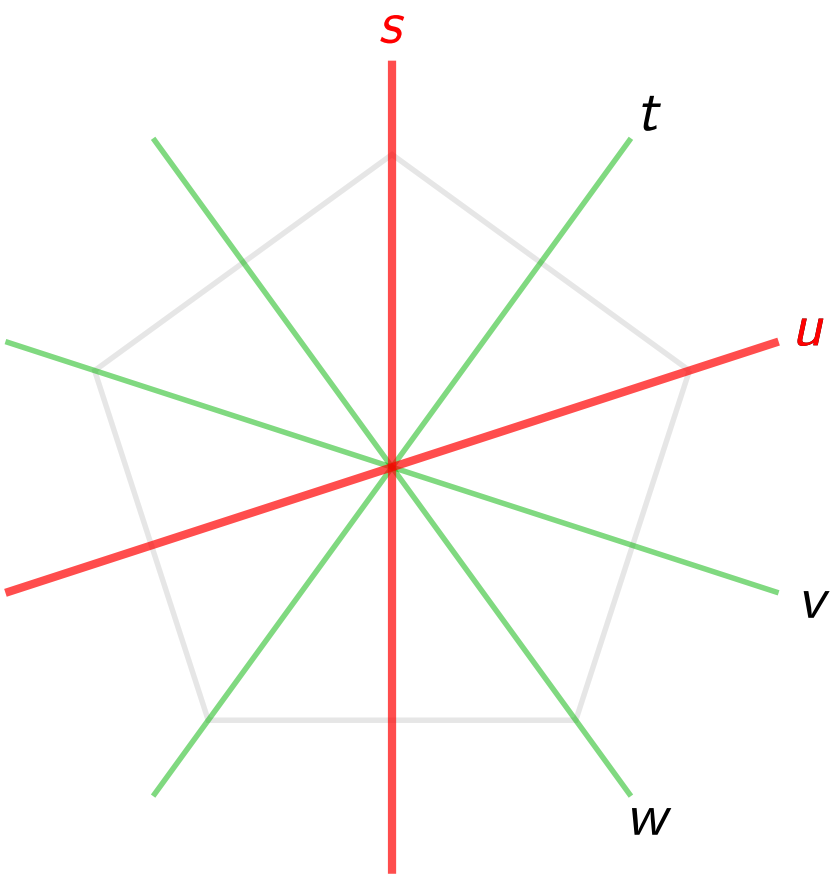

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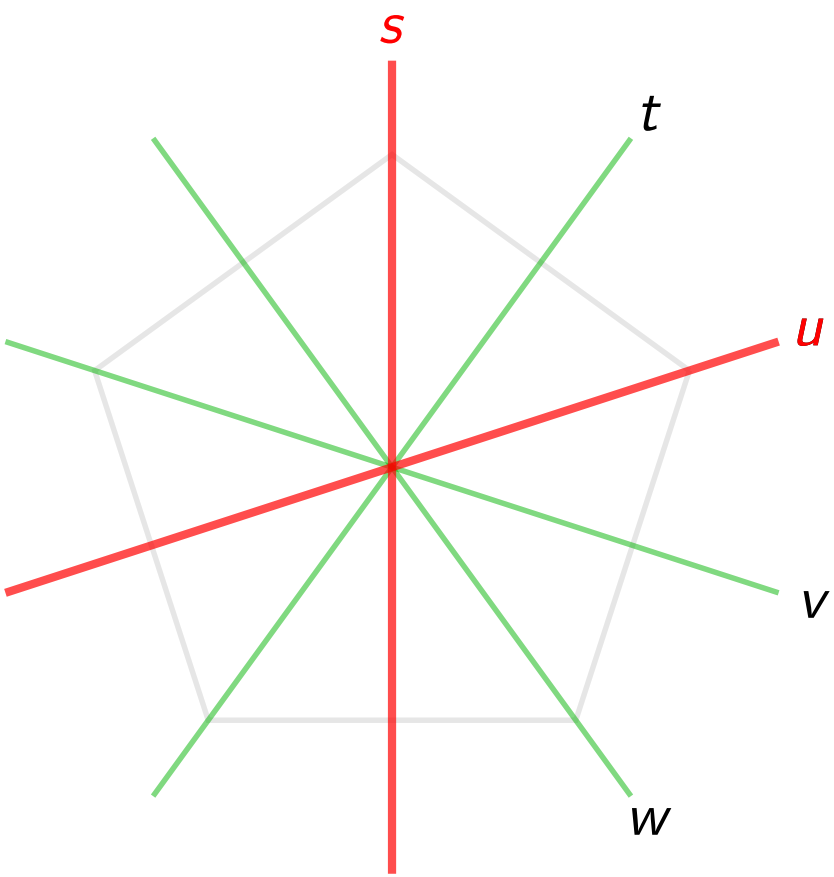

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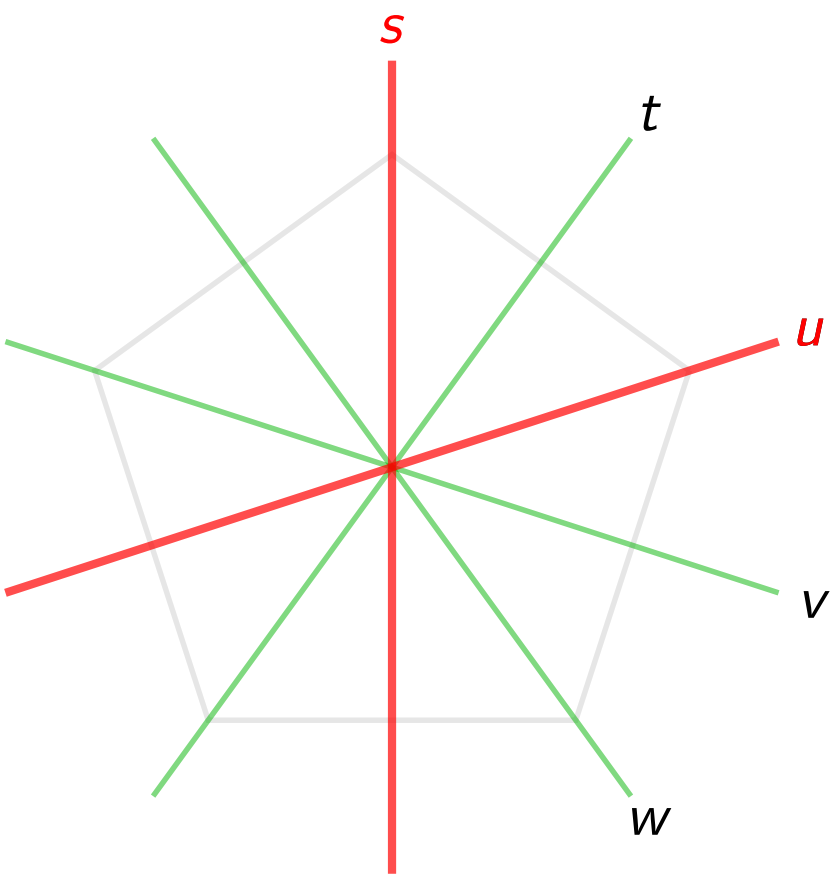

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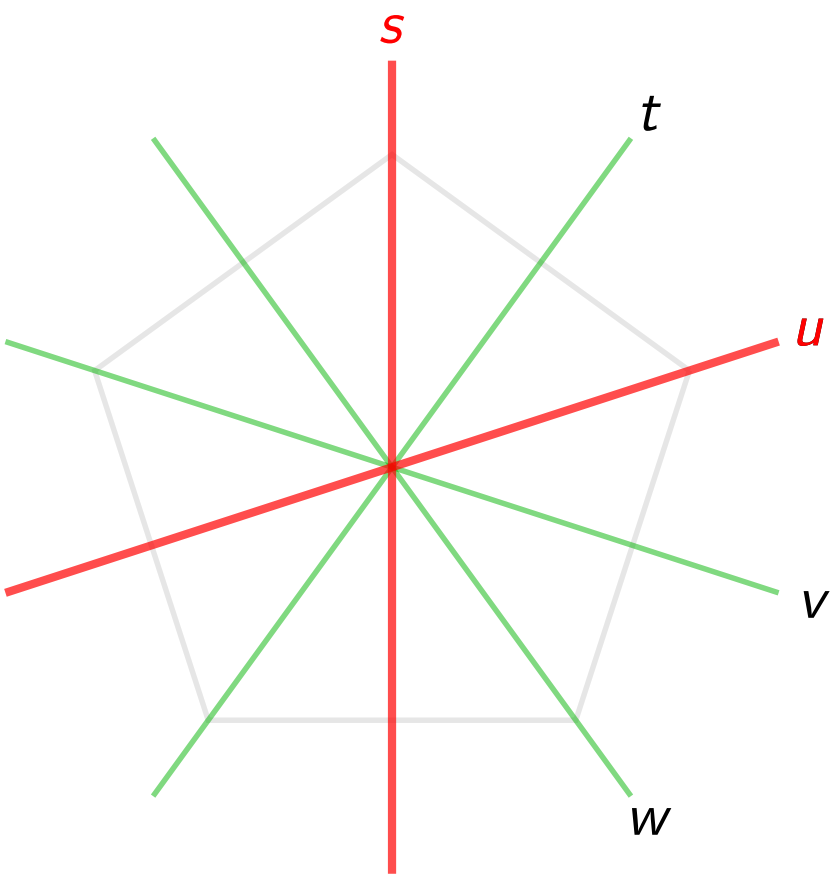

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New Coxeter elements

Definition (1.b.)

We call generalized Coxeter element of W a product (in any order) of the elements of some set S, where S is such that:

- S consists of reflections;
- (W, S) is a Coxeter system.

	Classical definition	Extended definition
W real	Product of reflections through the walls of a chamber	$\prod_{s \in S} s$ for some $S \subseteq R$, with (W, S) Coxeter
W complex	...	...

Rigidity of Coxeter structures

The new Coxeter elements have the same behaviour as the classical ones. Because the possible Coxeter structures (W, S), provided $S \subseteq R$, are all the "same":

Proposition (Observation/Folklore?)

Let W be a finite real reflection group, R the set of all reflections of W. Let $S, S' \subseteq R$ be such that (W, S) and (W, S') are both Coxeter systems.
Then (W, S) and (W, S') are isomorphic Coxeter systems.

In other words:

Let (W, S) be a **finite** Coxeter system, $R := \bigcup_{w \in W} wSw^{-1}$. Let $S' \subseteq R$ be such that (W, S') is also a Coxeter system.
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proof not enlightening! (case-by-case check on the classification)

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   - Classical definition
   - Extended definition
2. Coxeter elements in well-generated complex reflection groups — *via* eigenvalues
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Complex reflection groups

Finite *real* reflection groups can be seen as complex reflection groups.

But there are many more complex reflection groups.
In general: no Coxeter structure, no privileged (natural, canonical) set of n generating reflections.

Let W be a (well-generated) **complex** reflection group.
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Recall: Geometry of Coxeter elements in *real* groups

Assume W is a finite, **real** reflection group (irreducible). Let c be a classical (1.a.) Coxeter element of W, h the order of c (Coxeter number).

Facts

- $h = d_n$, the highest invariant degree of W;
  $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{C}[V]$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.
- The set of elements of W having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of W. [Springer's theory of regular elements]
- c admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

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c is a (classical) Coxeter element of W if and only if c admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

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Assume W is a finite, **real** reflection group (irreducible). Let c be a classical (1.a.) Coxeter element of W, h the order of c (Coxeter number).

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Almost done...

	Classical definition	Extended definition
1. W real	Product of reflections through the walls of a chamber	$\prod_{s \in S} s$, for some $S \subseteq R$, with (W, S) Coxeter
2. W complex	$e^{\frac{2i\pi}{h}}$ is eigenvalue	...

Replace $e^{2i\pi/h}$ by another h-th root of unity

Natural generalization: "Galois twist".

Definition ("Extended definition")

Let W be a well-generated, irreducible complex reflection group, and h its Coxeter number.

We call generalized Coxeter element an element of W that admits a primitive h-th root of unity as an eigenvalue.

Equivalently, c is a generalized Coxeter element if and only if $c = w^k$ where w is a *classical* Coxeter element and $k \wedge h = 1$.

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Four definitions!

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The two classical definitions are compatible.

Are the two extended definitions compatible as well?

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Outline

1. Coxeter elements in real reflection groups — *via* Coxeter systems
   - Classical definition
   - Extended definition
2. Coxeter elements in well-generated complex reflection groups — *via* eigenvalues
   - Classical definition
   - Extended definition
3. Reflection automorphisms and main results

Stability by reflection automorphisms

Definition

A reflection automorphism of W is an automorphism of W that preserves the set R of all reflections of W.

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) *c has an eigenvalue of order h;*

(ii) *$c = \psi(w)$ where w is a classical Coxeter element and ψ is a reflection automorphism of W;*

(iii) *(c is a Springer-regular element of order h).*

If W is real, this is also equivalent to:

(iv) *There exists $S \subseteq R$ such that (W, S) is a Coxeter system and c is the product (in any order) of elements of S.*

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Application to Coxeter-Catalan combinatorics

Let W be a well-generated, irreducible complex reflection group, and c a Coxeter element of W. Define the associated noncrossing partition lattice:

$$\mathrm{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\}$$

Corollary

For all generalized Coxeter elements c, the noncrossing partition lattices $\mathrm{NC}(W, c)$ are all isomorphic posets.

More generally, any property

- known for classical Coxeter elements, and
- depending only on the combinatorics of the couple (W, R),
- $\rightsquigarrow$ extends to generalized Coxeter elements.

Applies to properties related to Coxeter-Catalan combinatorics. For example, the number of reduced decompositions of a generalized Coxeter element into reflections is $\frac{n!\,h^n}{|W|}$.

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How many new Coxeter elements?

Definition

The field of definition K_W of W is the smallest field extension of $\mathbb{Q}$ over which one can write all matrices of complex W-representations.

Examples: $K_W = \mathbb{Q}$ iff W crystallographic (Weyl group).
For $W = I_2(m)$, $K_W = \mathbb{Q}(\cos \frac{2\pi}{m})$.

Theorem (Reiner-R.-Stump)

- *The number of conjugacy classes of generalized Coxeter elements is $[K_W : \mathbb{Q}]$.*
  (only 1 for Weyl groups; $\varphi(m)/2$ for dihedral group $I_2(m)$...)
- *More precisely, there is a natural action of the Galois group $\mathrm{Gal}(K_W/\mathbb{Q})$ on the set of conjugacy classes of generalized Coxeter elements of W, and this action is simply transitive.*

$$\forall C, C' \in \mathrm{Cox}(W), \exists \gamma \in \Gamma, C' = \gamma \cdot C.$$

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Ingredients of the proofs

- a spoonful of classical Springer's theory of regular elements
- a big chunk of Galois automorphisms and reflection automorphisms of W [Marin-Michel '10]
- a pinch of case-by-case checks ☹

Further results and questions

- Some results extend to more general elements of W, namely Springer's regular elements of arbitrary order.
- the characterization of generalized Coxeter elements for real groups extends to Shephard groups (those nicer complex groups with presentations "à la Coxeter").
- for the other well-generated complex groups, there is no canonical form of presentation, and not (yet?) a "combinatorial" vision of Coxeter elements.

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