Coxeter elements in well-generated reflection groups

Vivien Ripoll
(Universität Wien)

FPSAC/SFCA 2015
Daejeon, 2015, July 6th

joint work with
Vic Reiner (Minneapolis) and Christian Stump (Berlin)
Introduction

- $V$ vector space of dimension $n$, over $\mathbb{R}$ or $\mathbb{C}$

- $W$ finite subgroup of $\text{GL}(V)$ generated by reflections
  ($r \in \text{GL}(V)$ of finite order and fixing pointwise a hyperplane)

Finite real reflection groups correspond to finite Coxeter groups.

Complex reflection groups are more... complex. We consider only those that are well-generated, i.e., can be generated by $n$ reflections.
Introduction

- $V$ vector space of dimension $n$, over $\mathbb{R}$ or $\mathbb{C}$

- $W$ finite subgroup of $\text{GL}(V)$ generated by reflections ($r \in \text{GL}(V)$ of finite order and fixing pointwise a hyperplane)

Finite real reflection groups correspond to finite Coxeter groups.

Complex reflection groups are more... complex. We consider only those that are well-generated, i.e., can be generated by $n$ reflections.
Introduction

- $V$ vector space of dimension $n$, over $\mathbb{R}$ or $\mathbb{C}$

- $W$ finite subgroup of $\text{GL}(V)$ generated by reflections
  $(r \in \text{GL}(V)$ of finite order and fixing pointwise a hyperplane)

Finite real reflection groups correspond to finite Coxeter groups.

Complex reflection groups are more... complex.
We consider only those that are well-generated, i.e., can be
generated by $n$ reflections.
Introduction

- $R := \{\text{all reflections of } W\}$
- $\ell_R : W \to \mathbb{N}$ reflection length (or absolute length)

Coxeter elements are special elements $c$ of $W$ such that
- $\ell_R(c) = n$
- $c$ is product of a “nice” generating set of $n$ reflections.

Motivations:
- Coxeter-Catalan combinatorics
- understanding presentations for complex reflection groups

Questions:
- What is exactly a Coxeter element?
- How are different Coxeter elements related?
- Which role do these Coxeter elements play in the geometry and combinatorics of reflection groups?
Introduction

- $R := \{\text{all reflections of } W\}$
- $\ell_R : W \rightarrow \mathbb{N}$ reflection length (or absolute length)

**Coxeter elements** are special elements $c$ of $W$ such that
- $\ell_R(c) = n$
- $c$ is product of a “nice” generating set of $n$ reflections.

Motivations:
- Coxeter-Catalan combinatorics
- understanding presentations for complex reflection groups

Questions:
- What is exactly a Coxeter element?
- How are different Coxeter elements related?
- Which role do these Coxeter elements play in the geometry and combinatorics of reflection groups?
Introduction

- $R := \{\text{all reflections of } W\}$
- $\ell_R : W \to \mathbb{N}$ reflection length (or absolute length)

**Coxeter elements** are special elements $c$ of $W$ such that
  - $\ell_R(c) = n$
  - $c$ is product of a “nice” generating set of $n$ reflections.

**Motivations:**
  - Coxeter-Catalan combinatorics
  - understanding presentations for complex reflection groups

**Questions:**
  - What is exactly a Coxeter element?
  - How are different Coxeter elements related?
  - Which role do these Coxeter elements play in the geometry and combinatorics of reflection groups?
Introduction

- \( R := \{ \text{all reflections of } \mathcal{W} \} \)
- \( \ell_R : \mathcal{W} \to \mathbb{N} \) reflection length (or absolute length)

**Coxeter elements** are special elements \( c \) of \( \mathcal{W} \) such that
- \( \ell_R(c) = n \)
- \( c \) is product of a “nice” generating set of \( n \) reflections.

Motivations:
- Coxeter-Catalan combinatorics
- understanding presentations for complex reflection groups

Questions:
- What is exactly a Coxeter element?
- How are different Coxeter elements related?
- Which role do these Coxeter elements play in the geometry and combinatorics of reflection groups?
### Outline

<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>W real reflection group</td>
<td>1.a.</td>
</tr>
<tr>
<td></td>
<td>→ using Coxeter generators</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>W complex reflection group</td>
<td>2.a.</td>
</tr>
<tr>
<td></td>
<td>→ using eigenvalues</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Reflection automorphisms and main results</td>
<td></td>
</tr>
</tbody>
</table>
Outline

1. Coxeter elements in real reflection groups — via Coxeter systems
   - Classical definition
   - Extended definition

2. Coxeter elements in well-generated complex reflection groups — via eigenvalues
   - Classical definition
   - Extended definition

3. Reflection automorphisms and main results
Real reflection group and Coxeter system

Let $W$ be a **real** reflection group.

Choose a **chamber** $C$ of the hyperplane arrangement of $W$. Define

$$ S := \{ \text{reflections through the walls of } C \}.$$

Then:

- $S$ generates $W$;
- $(W, S)$ is a (finite) **Coxeter system**:

**Coxeter system**

The group $W$ has a presentation of the form:

$$ W = \langle S \mid s^2 = 1 \ (\forall s \in S); \ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \rangle,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2}$ for $s \neq t$. 
Let \( W \) be a real reflection group.

Choose a chamber \( C \) of the hyperplane arrangement of \( W \). Define

\[
S := \{\text{reflections through the walls of } C\}.
\]

Then:
- \( S \) generates \( W \);
- \((W, S)\) is a (finite) Coxeter system:

Coxeter system

The group \( W \) has a presentation of the form:

\[
W = \langle S \mid s^2 = 1 (\forall s \in S); (st)^{m_{s,t}} = 1 (\forall s \neq t \in S) \rangle,
\]

with \( m_{s,t} \in \mathbb{N}_{\geq 2} \) for \( s \neq t \).
Let $W$ be a real reflection group.

Choose a chamber $C$ of the hyperplane arrangement of $W$. Define

$$S := \{\text{reflections through the walls of } C\}.$$ 

Then:
- $S$ generates $W$;
- $(W, S)$ is a (finite) Coxeter system:

**Coxeter system**

The group $W$ has a presentation of the form:

$$W = \langle S \mid s^2 = 1 \ (\forall s \in S); \ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \rangle,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2}$ for $s \neq t$. 
Example: a dihedral group
Example: a dihedral group
Example: a dihedral group
Example: a dihedral group
Example: a dihedral group
Example: a dihedral group

\[ W = \langle s, t \mid s^2 = t^2 = 1, (st)^5 = 1 \rangle \]

Coxeter element = product of such generators, coming from the choice of a chamber:
- \( st, ts \)
- \( tu, ut \)
- \( uv, vu \)
- \( vw, wv \)
- \( ws, sw \)

\( \sim \) only two distinct Coxeter elements: rotations of angle \( \pm \frac{2\pi}{5} \).
Example: a dihedral group

\[ W = \langle s, t \mid s^2 = t^2 = 1, (st)^5 = 1 \rangle \]

Coxeter element = product of such generators, coming from the choice of a chamber:

- \( st, ts \)
- \( tu, ut \)
- \( uv, vu \)
- \( vw, wv \)
- \( ws, sw \)

\( \sim \) only two distinct Coxeter elements: rotations of angle \( \pm \frac{2\pi}{5} \).
Example: a dihedral group

\[ W = \langle s, t \mid s^2 = t^2 = 1, (st)^5 = 1 \rangle \]

Coxeter element = product of such generators, coming from the choice of a chamber:
- \( st, ts \)
- \( tu, ut \)
- \( uv, vu \)
- \( vw, wv \)
- \( ws, sw \)

\( \sim \) only two distinct Coxeter elements: rotations of angle \( \pm \frac{2\pi}{5} \).
Example: a dihedral group

\[ W = \langle s, t \mid s^2 = t^2 = 1, (st)^5 = 1 \rangle \]

Coxeter element = product of such generators, coming from the choice of a chamber:

- \( st, ts \)
- \( tu, ut \)
- \( uv, vu \)
- \( vw, wv \)
- \( ws, sw \)

\( \sim \) only two distinct Coxeter elements: rotations of angle \( \pm \frac{2\pi}{5} \).
Coxeter elements in a real reflection group

Definition (1.a.)
Let $W$ be a finite real reflection group. A **Coxeter element of $W$** is a product (in **any order**) of all the reflections through the walls of a **chamber** of $W$.

### Classical definition
- **$W$ real**
  - Product of reflections through the **walls of a chamber**

### Extended definition
- **$W$ complex**
  - ...
  - ...

**Proposition**

*These Coxeter elements form a conjugacy class of $W$.*
Coxeter elements in a real reflection group

Definition (1.a.)
Let $W$ be a finite real reflection group. A Coxeter element of $W$ is a product (in any order) of all the reflections through the walls of a chamber of $W$.

<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$ real</td>
<td>Product of reflections through the walls of a chamber</td>
<td>...</td>
</tr>
<tr>
<td>$W$ complex</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Proposition
These Coxeter elements form a conjugacy class of $W$. 
Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$

$\sim$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\sim$ Define more general Coxeter elements $=$ product of such generators

New Coxeter elements here: $su, us, tv, vt$...

$\sim$ two new elements: rotations of angle $\pm \frac{4\pi}{5}$. 
Dihedral group: unorthodox Coxeter elements

For a reflection group \( W \), there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take \( S = \{s, u\} \).

\[ W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle \]

\( \sim \) Coxeter structure isomorphic to the classical one, but not conjugate!

\( \sim \) Define more general Coxeter elements = product of such generators

New Coxeter elements here: \( su, us, tv, vt \ldots \)

\( \sim \) two new elements: rotations of angle \( \pm \frac{4\pi}{5} \).
Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$$

$\leadsto$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\leadsto$ Define more general Coxeter elements = product of such generators

New Coxeter elements here:

$su, us, tv, vt$...

$\leadsto$ two new elements:

rotations of angle $\pm \frac{4\pi}{5}$. 
Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$$

$\sim$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\sim$ Define more general Coxeter elements = product of such generators

New Coxeter elements here: $su, us, tv, vt$...

$\sim$ two new elements: rotations of angle $\pm \frac{4\pi}{5}$.
Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$$

$\leadsto$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\leadsto$ Define more general Coxeter elements = product of such generators

New Coxeter elements here: $su, us, tv, vt...$

$\leadsto$ two new elements: rotations of angle $\pm \frac{4\pi}{5}$. 
Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$$

$\leadsto$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\leadsto$ Define more general Coxeter elements $=$ product of such generators

New Coxeter elements here: $su, us, tv, vt$...

$\leadsto$ two new elements: rotations of angle $\pm \frac{4\pi}{5}$. 

Dihedral group: unorthodox Coxeter elements

For a reflection group $W$, there may be alternative Coxeter structures, which do not come from a chamber of the arrangement...

Here take $S = \{s, u\}$.

$W = \langle s, u \mid s^2 = u^2 = 1, (su)^5 = 1 \rangle$

$\sim$ Coxeter structure isomorphic to the classical one, but not conjugate!

$\sim$ Define more general Coxeter elements = product of such generators

New Coxeter elements here: $su, us, tv, vt$...

$\sim$ two new elements: rotations of angle $\pm \frac{4\pi}{5}$. 

\[ s, u, t, v, w \]
New Coxeter elements

Definition (1.b.)

We call generalized Coxeter element of $W$ a product (in any order) of the elements of some set $S$, where $S$ is such that:

- $S$ consists of reflections;
- $(W, S)$ is a Coxeter system.

<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$ real</td>
<td>Product of reflections through the walls of a chamber</td>
<td>$\prod_{s \in S} s$ for some $S \subseteq R$, with $(W, S)$ Coxeter</td>
</tr>
<tr>
<td>$W$ complex</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Rigidity of Coxeter structures

The new Coxeter elements have the same behaviour as the classical ones. Because the possible Coxeter structures \((W, S)\), provided \(S \subseteq R\), are all the “same”:

**Proposition (Observation/Folklore?)**

Let \(W\) be a finite real reflection group, \(R\) the set of all reflections of \(W\). Let \(S, S' \subseteq R\) be such that \((W, S)\) and \((W, S')\) are both Coxeter systems.

Then \((W, S)\) and \((W, S')\) are isomorphic Coxeter systems.

In other words:

Let \((W, S)\) be a finite Coxeter system, \(R := \bigcup_{w \in W} wSw^{-1}\). Let \(S' \subseteq R\) be such that \((W, S')\) is also a Coxeter system.

Then \((W, S')\) is isomorphic to \((W, S)\).

proof not enlightening! (case-by-case check on the classification)
Rigidity of Coxeter structures

The new Coxeter elements have the same behaviour as the classical ones. Because the possible Coxeter structures \((W, S)\), provided \(S \subseteq R\), are all the “same”:

**Proposition (Observation/Folklore?)**

Let \(W\) be a finite real reflection group, \(R\) the set of all reflections of \(W\). Let \(S, S' \subseteq R\) be such that \((W, S)\) and \((W, S')\) are both Coxeter systems. Then \((W, S)\) and \((W, S')\) are isomorphic Coxeter systems.

In other words:

Let \((W, S)\) be a finite Coxeter system, \(R := \bigcup_{w \in W} wSw^{-1}\). Let \(S' \subseteq R\) be such that \((W, S')\) is also a Coxeter system. Then \((W, S')\) is isomorphic to \((W, S)\).

Proof not enlightening! (case-by-case check on the classification)
Rigidity of Coxeter structures

The new Coxeter elements have the same behaviour as the classical ones. Because the possible Coxeter structures \((W, S)\), provided \(S \subseteq R\), are all the “same”:

**Proposition (Observation/Folklore?)**

Let \(W\) be a finite real reflection group, \(R\) the set of all reflections of \(W\). Let \(S, S' \subseteq R\) be such that \((W, S)\) and \((W, S')\) are both Coxeter systems. Then \((W, S)\) and \((W, S')\) are isomorphic Coxeter systems.

**In other words:**

Let \((W, S)\) be a finite Coxeter system, \(R := \bigcup_{w \in W} wSw^{-1}\). Let \(S' \subseteq R\) be such that \((W, S')\) is also a Coxeter system. Then \((W, S')\) is isomorphic to \((W, S)\).

proof not enlightening! (case-by-case check on the classification)
Rigidity of Coxeter structures

The new Coxeter elements have the same behaviour as the classical ones. Because the possible Coxeter structures \((W, S)\), provided \(S \subseteq R\), are all the “same”:

**Proposition (Observation/Folklore?)**

Let \(W\) be a finite real reflection group, \(R\) the set of all reflections of \(W\). Let \(S, S' \subseteq R\) be such that \((W, S)\) and \((W, S')\) are both Coxeter systems.

Then \((W, S)\) and \((W, S')\) are isomorphic Coxeter systems.

**In other words:**

Let \((W, S)\) be a finite Coxeter system, \(R := \bigcup_{w \in W} wSw^{-1}\). Let \(S' \subseteq R\) be such that \((W, S')\) is also a Coxeter system.

Then \((W, S')\) is isomorphic to \((W, S)\).

proof not enlightening! (case-by-case check on the classification)
Outline

1. Coxeter elements in real reflection groups — via Coxeter systems
   - Classical definition
   - Extended definition

2. Coxeter elements in well-generated complex reflection groups — via eigenvalues
   - Classical definition
   - Extended definition

3. Reflection automorphisms and main results
Finite real reflection groups can be seen as complex reflection groups.

But there are many more complex reflection groups. In general: no Coxeter structure, no privileged (natural, canonical) set of $n$ generating reflections.

Let $W$ be a (well-generated) complex reflection group. How to define a Coxeter element of $W$?
Finite *real* reflection groups can be seen as complex reflection groups.

But there are many more complex reflection groups. In general: no Coxeter structure, no privileged (natural, canonical) set of $n$ generating reflections.

Let $W$ be a (well-generated) *complex* reflection group. How to define a Coxeter element of $W$?
Finite *real* reflection groups can be seen as complex reflection groups.

But there are many more complex reflection groups. In general: no Coxeter structure, no privileged (natural, canonical) set of $n$ generating reflections.

Let $W$ be a (well-generated) *complex* reflection group. 

$\sim$ how to define a Coxeter element of $W$?
Recall: Geometry of Coxeter elements in real groups

Assume $W$ is a finite, real reflection group (irreducible). Let $c$ be a classical (1.a.) Coxeter element of $W$, $h$ the order of $c$ (Coxeter number).

Facts

- $h = d_n$, the highest invariant degree of $W$;
  - $d_1 \leq \ldots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n \in C[V]$ such that $C[V]^W = C[f_1, \ldots, f_n]$.
- The set of elements of $W$ having $e^{2i\pi h}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$. [Springer's theory of regular elements]
- $c$ admits $e^{2i\pi h}$ as an eigenvalue.

Proposition

$c$ is a (classical) Coxeter element of $W$ if and only if $c$ admits $e^{2i\pi h}$ as an eigenvalue.
Recall: Geometry of Coxeter elements in real groups

Assume $W$ is a finite, real reflection group (irreducible). Let $c$ be a classical (1.a.) Coxeter element of $W$, $h$ the order of $c$ (Coxeter number).

**Facts**

- $h = d_n$, the highest invariant degree of $W$: $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1$, $\ldots$, $f_n \in \mathbb{C}[V]$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.
- The set of elements of $W$ having $e^{2i\pi h}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$. [Springer’s theory of regular elements]
- $c$ admits $e^{2i\pi h}$ as an eigenvalue.

**Proposition**

$c$ is a (classical) Coxeter element of $W$ if and only if $c$ admits $e^{2i\pi h}$ as an eigenvalue.
Recall: Geometry of Coxeter elements in real groups

Assume $W$ is a finite, real reflection group (irreducible). Let $c$ be a classical (1.a.) Coxeter element of $W$, $h$ the order of $c$ (Coxeter number).

**Facts**

- $h = d_n$, the highest invariant degree of $W$:
  
  $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{C}[V]$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

- The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$. [Springer’s theory of regular elements]

- $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**Proposition**

$c$ is a (classical) Coxeter element of $W$ if and only if $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
Recall: Geometry of Coxeter elements in \textit{real} groups

Assume $W$ is a finite, \textit{real} reflection group (irreducible). Let $c$ be a classical (1.a.) Coxeter element of $W$, $h$ the order of $c$ (Coxeter number).

### Facts

- $h = d_n$, the highest invariant degree of $W$: $d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{C}[V]$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.
- The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$. [Springer’s theory of regular elements]
- $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

### Proposition

$c$ is a (classical) Coxeter element of $W$ if and only if $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
Recall: Geometry of Coxeter elements in real groups

Assume \( W \) is a finite, real reflection group (irreducible). Let \( c \) be a classical (1.a.) Coxeter element of \( W \), \( h \) the order of \( c \) (Coxeter number).

**Facts**

- \( h = d_n \), the highest invariant degree of \( W \):
  
  \[ d_1 \leq \cdots \leq d_n \]

  degrees of homogeneous polynomials

  \[ f_1, \ldots, f_n \in \mathbb{C}[V] \text{ such that } \mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]. \]

- The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \). [Springer’s theory of regular elements]

- \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.

**Proposition**

\( c \) is a (classical) Coxeter element of \( W \) if and only if \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.
Recall: Geometry of Coxeter elements in real groups

Assume \( W \) is a finite, real reflection group (irreducible). Let \( c \) be a classical (1.a.) Coxeter element of \( W \), \( h \) the order of \( c \) (Coxeter number).

**Facts**

- \( h = d_n \), the highest invariant degree of \( W \):
  - \( d_1 \leq \cdots \leq d_n \) degrees of homogeneous polynomials
  - \( f_1, \ldots, f_n \in \mathbb{C}[V] \) such that \( \mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n] \).

- The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \). [Springer’s theory of regular elements]

- \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.

**Proposition**

\( c \) is a (classical) Coxeter element of \( W \) if and only if \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.
Geometry of Coxeter elements

**$W$ real (irreducible)**

$c$ a Coxeter element, $h$ its order

- $h = d_n$, highest invariant degree
- The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
  [Springer’s theory of regular elements]
- $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**$W$ complex**

Assume $W$ is well-generated.

- Define $h = d_n$.
- The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
  [Springer’s theory]

**Definition (2.a.) (Bessis ’06)**

Call Coxeter element of $W$ an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**Proposition**

$c$ is a **Coxeter element** of $W$ if and only if $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
**Geometry of Coxeter elements**

**W real (irreducible)**
- $c$ a Coxeter element, $h$ its order
  - $h = d_n$, highest invariant degree
  - The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
    - [Springer’s theory of regular elements]
  - $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**W complex**
- Assume $W$ is well-generated.
  - Define $h := d_n$.
  - The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
    - [Springer’s theory]

**Definition (2.a.) (Bessis ’06)**
- Call Coxeter element of $W$ an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**Proposition**
- $c$ is a Coxeter element of $W$ if and only if $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
### Geometry of Coxeter elements

#### \( W \) real (irreducible)
- \( c \) a Coxeter element, \( h \) its order
  - \( h = d_n \), highest invariant degree
  - The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \).
    - [Springer’s theory of regular elements]
  - \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.

#### \( W \) complex
- Assume \( W \) is well-generated.
  - Define \( h := d_n \).
  - The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \).
    - [Springer’s theory]

### Proposition
- \( c \) is a Coxeter element of \( W \) if and only if \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.
Geometry of Coxeter elements

\( W \text{ real (irreducible)} \)

- \( c \) a Coxeter element, \( h \) its order
  - \( h = d_n \), highest invariant degree
  - The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \).
    [Springer’s theory of regular elements]
  - \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.

\( W \text{ complex} \)

- Assume \( W \) is well-generated.
  - Define \( h := d_n \).
  - The set of elements of \( W \) having \( e^{\frac{2i\pi}{h}} \) as an eigenvalue is non-empty and forms a conjugacy class of \( W \).
    [Springer’s theory]

**Proposition**

\( c \) is a **Coxeter element** of \( W \) if and only if \( c \) admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.

Definition (2.a.) (Bessis ’06)

Call Coxeter element of \( W \) an element that admits \( e^{\frac{2i\pi}{h}} \) as an eigenvalue.
Geometry of Coxeter elements

**$W$ real (irreducible)**

- $c$ a Coxeter element, $h$ its order
  - $h = d_n$, highest invariant degree
  - The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
    [Springer’s theory of regular elements]
  - $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**$W$ complex**

- Assume $W$ is well-generated.
  - Define $h := d_n$.
  - The set of elements of $W$ having $e^{\frac{2i\pi}{h}}$ as an eigenvalue is non-empty and forms a conjugacy class of $W$.
    [Springer’s theory]

**Proposition**

- $c$ is a Coxeter element of $W$ if and only if $c$ admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

**Definition (2.a.) (Bessis ’06)**

- Call Coxeter element of $W$ an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.
<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. $W$ real</strong></td>
<td>Product of reflections through the walls of a chamber</td>
<td>$\prod_{s \in S} s$, for some $S \subseteq R$, with $(W, S)$ Coxeter</td>
</tr>
<tr>
<td><strong>2. $W$ complex</strong></td>
<td>$e^{\frac{2i\pi}{\hbar}}$ is eigenvalue</td>
<td>...</td>
</tr>
</tbody>
</table>
Natural generalization: “Galois twist”.

**Definition (“Extended definition”)**

Let $W$ be a well-generated, irreducible complex reflection group, and $h$ its Coxeter number.

We call generalized Coxeter element an element of $W$ that admits a primitive $h$-th root of unity as an eigenvalue.

Equivalently, $c$ is a generalized Coxeter element if and only if $c = w^k$ where $w$ is a classical Coxeter element and $k \wedge h = 1$. 

Replace $e^{2i\pi/h}$ by another $h$-th root of unity
Replace $e^{2i\pi/h}$ by another $h$-th root of unity

Natural generalization: “Galois twist”.

**Definition ("Extended definition")**

Let $W$ be a well-generated, irreducible complex reflection group, and $h$ its Coxeter number.

We call **generalized Coxeter element** an element of $W$ that admits a **primitive $h$-th root of unity** as an eigenvalue.

Equivalently, $c$ is a generalized Coxeter element if and only if $c = w^k$ where $w$ is a **classical** Coxeter element and $k \wedge h = 1$. 
Four definitions!

<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $W$ real</td>
<td>Product of reflections through the walls of a chamber</td>
<td>$\prod_{s \in S} s$, for some $S \subseteq R$, with $(W, S)$ Coxeter</td>
</tr>
<tr>
<td>2. $W$ complex</td>
<td>$e^{\frac{2i\pi}{h}}$ is eigenvalue</td>
<td>$e^{\frac{2ik\pi}{h}}$ is eigenvalue for some $k$, $k \wedge h = 1$</td>
</tr>
</tbody>
</table>

The two classical definitions are compatible.

Are the two extended definitions compatible as well?
Four definitions!

<table>
<thead>
<tr>
<th></th>
<th>Classical definition</th>
<th>Extended definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $W$ real</td>
<td>Product of reflections through the \textit{walls of a chamber}</td>
<td>$\prod_{s \in S} s$, for some $S \subseteq R$, with $(W, S)$ Coxeter</td>
</tr>
<tr>
<td>2. $W$ complex</td>
<td>$e^{\frac{2i\pi}{\hbar}}$ is eigenvalue</td>
<td>$e^{\frac{2ik\pi}{\hbar}}$ is eigenvalue for some $k$, $k \wedge h = 1$</td>
</tr>
</tbody>
</table>

The two classical definitions are compatible.
Are the two extended definitions compatible as well?
Outline

1. Coxeter elements in real reflection groups — via Coxeter systems
   - Classical definition
   - Extended definition

2. Coxeter elements in well-generated complex reflection groups — via eigenvalues
   - Classical definition
   - Extended definition

3. Reflection automorphisms and main results
Stability by reflection automorphisms

**Definition**
A reflection automorphism of \( W \) is an automorphism of \( W \) that preserves the set \( R \) of all reflections of \( W \).

**Theorem (Reiner-R.-Stump)**
Let \( c \in W \). The following are equivalent:

(i) \( c \) has an eigenvalue of order \( h \);

(ii) \( c = \psi(w) \) where \( w \) is a classical Coxeter element and \( \psi \) is a reflection automorphism of \( W \);

(iii) \( c \) is a Springer-regular element of order \( h \).

If \( W \) is real, this is also equivalent to:

(iv) There exists \( S \subseteq R \) such that \((W, S)\) is a Coxeter system and \( c \) is the product (in any order) of elements of \( S \).
Stability by reflection automorphisms

**Definition**

A reflection automorphism of $W$ is an automorphism of $W$ that preserves the set $R$ of all reflections of $W$.

**Theorem (Reiner-R.-Stump)**

Let $c \in W$. The following are equivalent:

1. *c has an eigenvalue of order $h$;*
2. $c = \psi(w)$ where $w$ is a classical Coxeter element and $\psi$ is a reflection automorphism of $W$;
3. ($c$ is a Springer-regular element of order $h$).

If $W$ is real, this is also equivalent to:

4. There exists $S \subseteq R$ such that $(W, S)$ is a Coxeter system and $c$ is the product (in any order) of elements of $S$. 
Definition

A reflection automorphism of $W$ is an automorphism of $W$ that preserves the set $R$ of all reflections of $W$.

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) $c$ has an eigenvalue of order $h$;

(ii) $c = \psi(w)$ where $w$ is a classical Coxeter element and $\psi$ is a reflection automorphism of $W$;

(iii) $(c$ is a Springer-regular element of order $h$).

If $W$ is real, this is also equivalent to:

(iv) There exists $S \subseteq R$ such that $(W, S)$ is a Coxeter system and $c$ is the product (in any order) of elements of $S$. 
Stability by reflection automorphisms

Definition

A reflection automorphism of $W$ is an automorphism of $W$ that preserves the set $R$ of all reflections of $W$.

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) $c$ has an eigenvalue of order $h$;
(ii) $c = \psi(w)$ where $w$ is a classical Coxeter element and $\psi$ is a reflection automorphism of $W$;
(iii) ($c$ is a Springer-regular element of order $h$).

If $W$ is real, this is also equivalent to:

(iv) There exists $S \subseteq R$ such that $(W, S)$ is a Coxeter system and $c$ is the product (in any order) of elements of $S$. 
Stability by reflection automorphisms

**Definition**
A reflection automorphism of $W$ is an automorphism of $W$ that preserves the set $R$ of all reflections of $W$.

**Theorem (Reiner-R.-Stump)**
Let $c \in W$. The following are equivalent:

(i) $c$ has an eigenvalue of order $h$;

(ii) $c = \psi(w)$ where $w$ is a classical Coxeter element and $\psi$ is a reflection automorphism of $W$;

(iii) ($c$ is a Springer-regular element of order $h$).

If $W$ is real, this is also equivalent to:

(iv) There exists $S \subseteq R$ such that $(W, S)$ is a Coxeter system and $c$ is the product (in any order) of elements of $S$. 


Definition

A **reflection automorphism of** \( W \) **is an automorphism of** \( W \) **that preserves the set** \( R \) **of all reflections of** \( W \).

Theorem (Reiner-R.-Stump)

Let \( c \in W \). The following are equivalent:

(i) \( c \) has an **eigenvalue of order** \( h \);

(ii) \( c = \psi(w) \) where \( w \) is a classical Coxeter element and \( \psi \) is a reflection automorphism of \( W \);

(iii) \( c \) is a Springer-regular element of order \( h \).

If \( W \) is **real**, this is also equivalent to:

(iv) There exists \( S \subseteq R \) such that \((W, S)\) is a Coxeter system and \( c \) is the product (in any order) of elements of \( S \).
Stability by reflection automorphisms

Definition
A reflection automorphism of $\mathcal{W}$ is an automorphism of $\mathcal{W}$ that preserves the set $R$ of all reflections of $\mathcal{W}$.

Theorem (Reiner-R.-Stump)
Let $c \in \mathcal{W}$. The following are equivalent:
(i) $c$ has an eigenvalue of order $h$;
(ii) $c = \psi(w)$ where $w$ is a classical Coxeter element and $\psi$ is a reflection automorphism of $\mathcal{W}$;
(iii) ($c$ is a Springer-regular element of order $h$).
If $\mathcal{W}$ is real, this is also equivalent to:
(iv) There exists $S \subseteq R$ such that $(\mathcal{W}, S)$ is a Coxeter system and $c$ is the product (in any order) of elements of $S$. 
Application to Coxeter-Catalan combinatorics

Let $W$ be a well-generated, irreducible complex reflection group, and $c$ a Coxeter element of $W$. Define the associated noncrossing partition lattice:

$$\text{NC}(W, c) := \{ w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c) \}$$

Corollary

For all generalized Coxeter elements $c$, the noncrossing partition lattices $\text{NC}(W, c)$ are all isomorphic posets.

More generally, any property

- known for classical Coxeter elements, and
- depending only on the combinatorics of the couple $(W, R)$,
- $\sim$ extends to generalized Coxeter elements.

Applies to properties related to Coxeter-Catalan combinatorics. For example, the number of reduced decompositions of a generalized Coxeter element into reflections is $\frac{n! h^n}{|W|}$. 
Application to Coxeter-Catalan combinatorics

Let $W$ be a well-generated, irreducible complex reflection group, and $c$ a Coxeter element of $W$. Define the associated noncrossing partition lattice:

$$\text{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\}$$

**Corollary**

For all generalized Coxeter elements $c$, the noncrossing partition lattices $\text{NC}(W, c)$ are all isomorphic posets.

More generally, any property

- known for classical Coxeter elements, and
- depending only on the combinatorics of the couple $(W, R)$,
- $\sim$ extends to generalized Coxeter elements.

Applies to properties related to Coxeter-Catalan combinatorics. For example, the number of reduced decompositions of a generalized Coxeter element into reflections is $\frac{n!h^n}{|W|}$. 
Application to Coxeter-Catalan combinatorics

Let $W$ be a well-generated, irreducible complex reflection group, and $c$ a Coxeter element of $W$. Define the associated noncrossing partition lattice:

$$\text{NC}(W, c) := \{ w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c) \}$$

**Corollary**

For all generalized Coxeter elements $c$, the noncrossing partition lattices $\text{NC}(W, c)$ are all isomorphic posets.

More generally, any property

- known for classical Coxeter elements, and
- depending only on the combinatorics of the couple $(W, R)$,
- $\sim$ extends to generalized Coxeter elements.

Applies to properties related to Coxeter-Catalan combinatorics. For example, the number of reduced decompositions of a generalized Coxeter element into reflections is $\frac{n! h^n}{|W|}$. 
How many new Coxeter elements?

**Definition**

The **field of definition** $K_W$ of $W$ is the smallest field extension of $\mathbb{Q}$ over which one can write all matrices of complex $W$-representations.

**Examples:** $K_W = \mathbb{Q}$ iff $W$ crystallographic (Weyl group).
For $W = I_2(m)$, $K_W = \mathbb{Q}(\cos \frac{2\pi}{m})$.

**Theorem (Reiner-R.-Stump)**

- The number of conjugacy classes of generalized Coxeter elements is $[K_W : \mathbb{Q}]$.
  (only 1 for Weyl groups; $\varphi(m)/2$ for dihedral group $I_2(m)$ ...)
- More precisely, there is a natural action of the Galois group $\text{Gal}(K_W/\mathbb{Q})$ on the set of conjugacy classes of generalized Coxeter elements of $W$, and this action is simply transitive.

\[ \forall C, C' \in \text{Cox}(W), \exists ! \gamma \in \Gamma : C' = \gamma \cdot C. \]
How many new Coxeter elements?

**Definition**

The **field of definition** $K_W$ of $W$ is the smallest field extension of $\mathbb{Q}$ over which one can write all matrices of complex $W$-representations.

**Examples:** $K_W = \mathbb{Q}$ iff $W$ crystallographic (Weyl group).

For $W = I_2(m)$, $K_W = \mathbb{Q}(\cos \frac{2\pi}{m})$.

**Theorem (Reiner-R.-Stump)**

- The number of **conjugacy classes** of generalized Coxeter elements is $[K_W : \mathbb{Q}]$.
  (only 1 for Weyl groups; $\varphi(m)/2$ for dihedral group $I_2(m)$...)

- More precisely, there is a natural action of the Galois group $\text{Gal}(K_W/\mathbb{Q})$ on the set of conjugacy classes of generalized Coxeter elements of $W$, and this action is simply transitive.

$$\forall C, C' \in \text{Cox}(W), \exists! \gamma \in \Gamma, C' = \gamma \cdot C.$$
How many new Coxeter elements?

**Definition**

The field of definition $K_W$ of $W$ is the smallest field extension of $\mathbb{Q}$ over which one can write all matrices of complex $W$-representations.

**Examples:** $K_W = \mathbb{Q}$ iff $W$ crystallographic (Weyl group). For $W = I_2(m)$, $K_W = \mathbb{Q}(\cos \frac{2\pi}{m})$.

**Theorem (Reiner-R.-Stump)**

- The number of conjugacy classes of generalized Coxeter elements is $[K_W : \mathbb{Q}]$.
  (only 1 for Weyl groups; $\varphi(m)/2$ for dihedral group $I_2(m)$...)

- More precisely, there is a natural action of the Galois group $\text{Gal}(K_W/\mathbb{Q})$ on the set of conjugacy classes of generalized Coxeter elements of $W$, and this action is simply transitive.
  
  $$\forall C, C' \in \text{Cox}(W), \exists! \gamma \in \Gamma, C' = \gamma \cdot C.$$
Ingredients of the proofs

- a spoonful of classical Springer’s theory of regular elements

- a big chunk of Galois automorphisms and reflection automorphisms of $W$ [Marin-Michel ’10]

- a pinch of case-by-case checks 😞
Further results and questions

- Some results extend to more general elements of $W$, namely Springer’s regular elements of arbitrary order.

- the characterization of generalized Coxeter elements for real groups extends to Shephard groups (those nicer complex groups with presentations “à la Coxeter”).

- for the other well-generated complex groups, there is no canonical form of presentation, and not (yet?) a “combinatorial” vision of Coxeter elements.
Further results and questions

- Some results extend to more general elements of $W$, namely Springer’s regular elements of arbitrary order.

- The characterization of generalized Coxeter elements for real groups extends to Shephard groups (those nicer complex groups with presentations “à la Coxeter”).

- For the other well-generated complex groups, there is no canonical form of presentation, and not (yet?) a “combinatorial” vision of Coxeter elements.

Thank you!