Quadrangulations, Stokes posets and serpent nests

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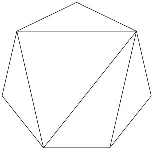
CNRS & Université Claude Bernard Lyon 1

July 2015

Triangulations

Start with classical objects: **triangulations** of regular polygons, already considered by Leonhard Euler.



a triangulation of an heptagon
chosen from the 42 possible ones

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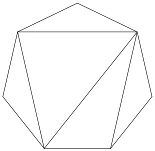
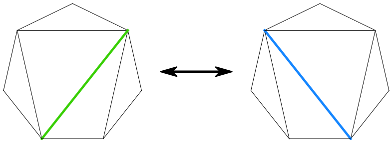


a triangulation of an heptagon
chosen from the 42 possible ones

The number of triangulations in the polygon with $n + 2$ sides is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.
A famous sequence of numbers, named after Eugène Catalan.

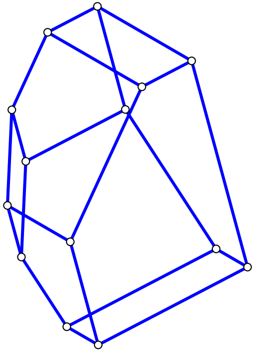
Flips between triangulations

Triangulations can be connected by **flips**,
replacing just one interior edge by another one:



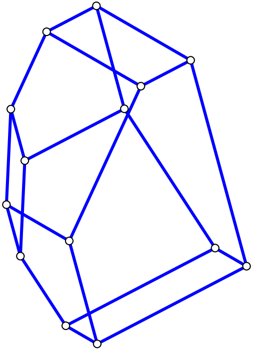
This gives a regular graph, the **flip graph** of triangulations.

Triangulations and associahedra



the flip graph of triangulations of the hexagon
14 vertices $\longleftrightarrow$ 14 triangulations
Edges are flips

Triangulations and associahedra



the flip graph of triangulations of the hexagon
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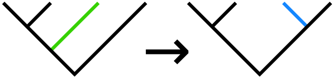
Edges are flips

It is also known that the flip graph can be realized using vertices and edges of a polytope called the **associahedra**, introduced by Jim Stasheff.

Oriented flips and Tamari lattices

By a standard bijection,

- triangulations $\leftrightarrow$ planar binary trees,
- flips $\leftrightarrow$ "rotation" of trees.



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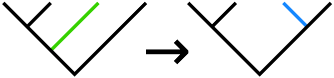
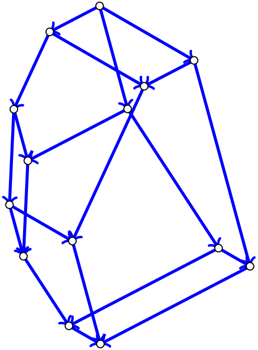


This allows to orient the edges of the flip graph, into the Hasse diagram of a poset, the **Tamari lattice** (Dov Tamari).



the Tamari lattice of triangulations of the hexagon
14 vertices $\longleftrightarrow$ 14 triangulations
$\longleftrightarrow$ 14 planar binary trees
Edges are flips
oriented from top to bottom.

Modern point of view: cluster combinatorics

All these objects are now considered to find a natural context in the theory of **cluster algebras**.

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Finite Coxeter group $W \longrightarrow$ Cambrian lattices for W

such that the special case of type $\mathbb{A}$ is

Symmetric group $\longrightarrow$ Tamari lattice (and other lattices)

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There are also polytopes called generalized associahedra.

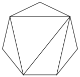
A trilogy for each integer n

Back to some other classical combinatorial objects.

Three classical families of objects counted by the Catalan numbers.

A triangulations

flip graph, lattice, polytope



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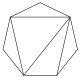
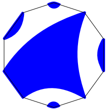


(B) noncrossing partitions (Germain Kreweras)

graded lattice



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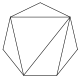

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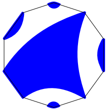

(C) Dyck paths (named after Walther von Dyck)

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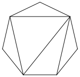
flip graph, lattice, polytope $\rightarrow$ F-triangle counting faces



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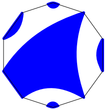
graded lattice $\rightarrow$ M-triangle for Möbius numbers



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distributive lattice $\rightarrow$ H-triangle

They are also tied by more refined enumerative properties.
Each triangle is a two-variable generating polynomial ; they are related by rational change-of-variables transformations.

F, M, H triangles

An example, for concreteness

F-triangle for triangulations of the pentagon:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 2 \end{pmatrix}$$

M-triangle for the noncrossing partitions lattice of size 5:

$$\begin{pmatrix} 2 & -3 & 1 \\ -3 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

H-triangle for Dyck paths with 3 up and 3 down steps:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

A trilogy for each Weyl group W

All three of them are now understood as "being of type $\mathbb{A}$"
and have been generalized to all finite Weyl groups,
some even to finite Coxeter groups or complex reflexion groups.

(A) triangulations $\rightarrow$ W-clusters

flip graph, Cambrian lattices, polytopes

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with still the same enumerative relations between
F-triangle, M-triangle and H-triangle.

Today: a brand new generalization!

more precisely, previous construction was
a Coxeter element c in Coxeter group $W \to$ a Cambrian lattice

The pair (W, c) is being replaced by a **quadrangulation** Q

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Ⓐ triangulations or W-clusters $\to$ Q-compatible quadrangulations

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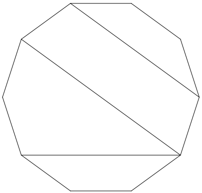
Again (A) and (C) are expected to have the same cardinality and the same enumerative relations between F-triangle and H-triangle.

Quadrangulations of regular polygons



set of lines between vertices cutting the polygon into parts with 4 sides

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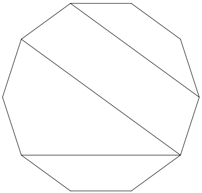

set of lines between vertices cutting the polygon into parts with 4 sides

The number of sides of the polygon must be even, say $2n+2$.
The number of quadrangulations is then $\frac{1}{2n+1}\binom{3n}{n}$,
called a Fuss-Catalan number (named after Nikolaus Fuss).

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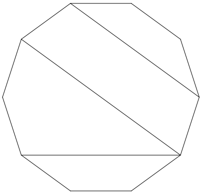

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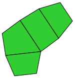
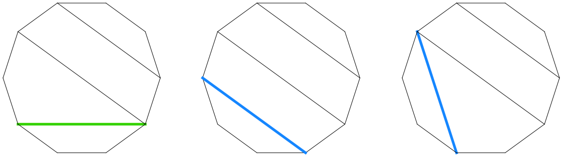
Quadrangulations can also be depicted like that: 
as a tree-like union of quadrilaterals along their edges.

Quadrangulations and ambiguous flips

One can flip quadrangulations, but there are two ways to replace any given edge.



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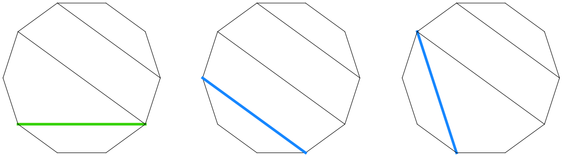

This has something to do with 2-cluster categories and 2-Cambrian lattices. Not exactly the subject of this talk.

Stokes polytopes

Yuliy Baryshnikov has defined, for every quadrangulation Q
a polytope called the **Stokes polytope**.
Some of them are associahedra!
His motivation came from the study of bifurcation diagrams of quadratic differentials (singularity theory, geometry).

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His motivation came from the study of bifurcation diagrams of quadratic differentials (singularity theory, geometry).
To define these polytopes, he introduced a **compatibility** relation between quadrangulations.

Theorem (Baryshnikov)

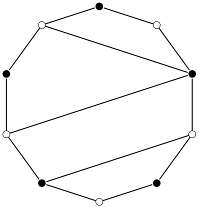
Let Q be a quadrangulation. There exists a polytope St_Q with

- *vertices $\longleftrightarrow$ Q-compatible quadrangulations,*
- *edges $\longleftrightarrow$ flips between them.*

For Q in the $2n+2$-sided polygon, the dimension of St_Q is $n-1$.

Compatibility of quadrangulations

Let Q be fixed. Let us now describe compatibility with Q.



- color vertices of Q by alternating black and white,

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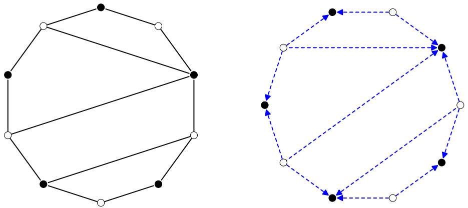

- color vertices of Q by alternating black and white,
- rotate Q by an angle of $\frac{2\pi}{4n+4}$ and color it blue,
- orient all edges of Q from white vertices $\circ$ to black vertices $\bullet$.

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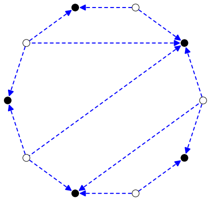
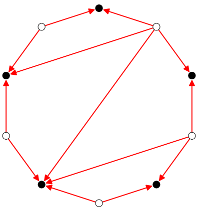


Q



Q'

- consider another quadrangulation Q' (color it red)
- color vertices of Q' by alternating black and white as before
- orient all edges of Q' from white vertices $\circ$ to black vertices $\bullet$.

Compatibility of quadrangulations

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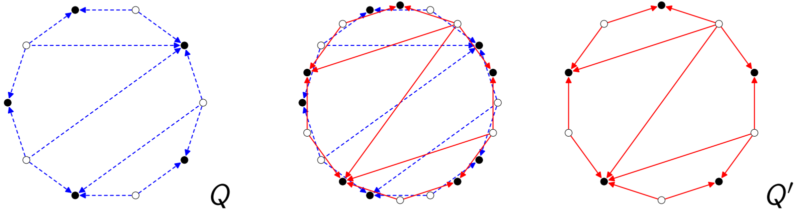
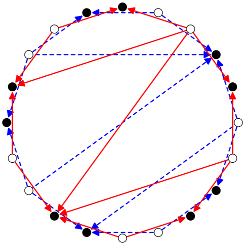


- consider another quadrangulation Q' (color it red)
- color vertices of Q' by alternating black and white as before
- orient all edges of Q' from white vertices $\circ$ to black vertices $\bullet$.
- superpose rotated Q and non-rotated Q'

Compatibility: at every crossing, $(\overrightarrow{red}, \overrightarrow{blue})$ has orientation $\circlearrowleft$

Q-compatible quadrangulations



No need to look closely at the boundary:
compatibility only needs to be checked at interior crossings.

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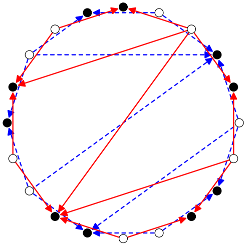

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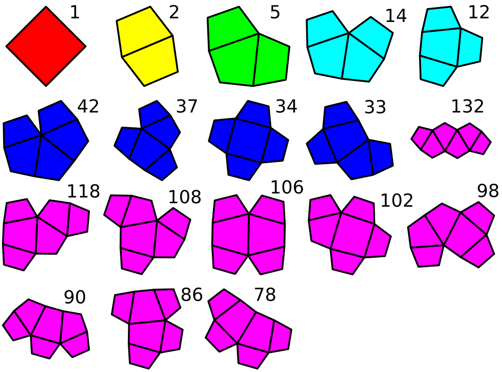
There are always at least two Q-compatible quadrangulations:
Q itself (not rotated) and Q rotated by $\frac{2\pi}{2n+2}$.

Q-compatible quadrangulations

The number of Q-compatible quadrangulations depends on Q:



Not a full table. Distinct Q can have the same number.

Flips of Q-compatible quadrangulations

Let Q be fixed.

Statement

Let Q' be a Q-compatible quadrangulation. Given any edge e of Q', there exists a unique other edge e' such that $Q - e + e'$ is a Q-compatible quadrangulation.

So one can always flip, and without having to choose!
Exactly one of the two possible flips is allowed.

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This gives a regular graph St_Q of Q-compatible quadrangulations.

This is the graph of edges and vertices of the Stokes polytopes.

Oriented flips of Q-compatible quadrangulations

One can in fact orient the flips in a natural way
and get a directed graph $\overrightarrow{\mathrm{St}}_Q$.

Theorem (C.)

The directed graph $\overrightarrow{\mathrm{St}}_Q$ is the Hasse diagram of a poset.

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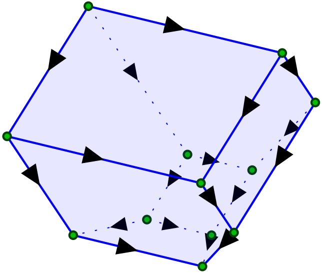

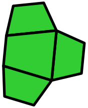

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The oriented flip graph on the 12 Q-compatible quadrangulations

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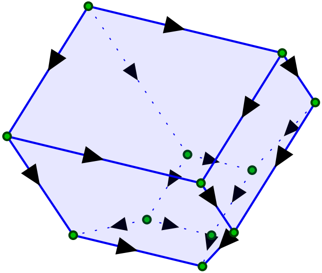

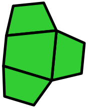

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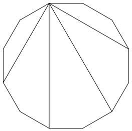
Conjecturally, all these posets are lattices.

Tamari lattice as a special case

One finds the Tamari lattices for the following quadrangulations



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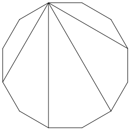

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It is moreover expected that one can also recover all the Cambrian lattices of type $\mathbb{A}$, from appropriate ("ribbon") quadrangulations.

Here comes the second part

Next step, the other side of the story

or why serpents are never crossing bridges

Ⓐ Q-compatible quadrangulations (analogs of triangulations)

flip graph, posets, polytopes,

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(A) Q-compatible quadrangulations (analogs of triangulations)

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(C) serpent nests in Q (analogs of Dyck paths)

graded set with duality

Next step, the other side of the story

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(A) Q-compatible quadrangulations (analogs of triangulations)

flip graph, posets, polytopes, fans, toric varieties

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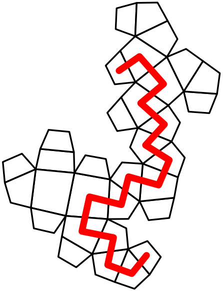
This is supposed to go that way: from polytopes one can go to fans and toric varieties.
The other side of the story is supposed to be related to the cohomology of the toric variety.
Just a motivation, no clear statement so far.

Serpent nests: serpent

Serpent = another word for snake

Fix a background quadrangulation Q.

A **serpent** (in Q) is a path joining two square centers (with steps at square centers) and turning either left or right at every step



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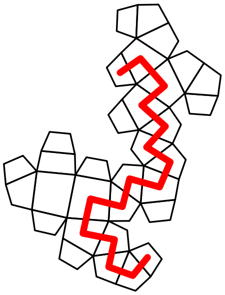
Serpent = another word for snake

Fix a background quadrangulation Q.

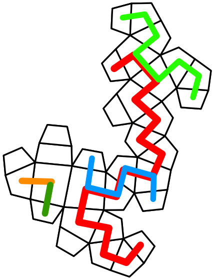
A **serpent** (in Q) is a path joining two square centers (with steps at square centers) and turning either left or right at every step



Never cross a square by going straight to the opposite side!

Serpent nests: definition

A **serpent nest** is a set of serpents + some conditions and modulo some equivalence relation



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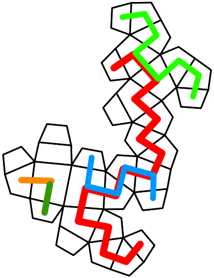

Condition: no two ends can share both the same square center and the same exit side:

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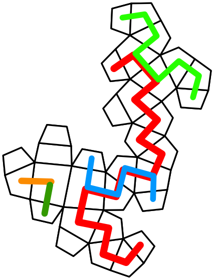

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Equivalence: at every edge of Q, one can change arbitrarily the connections between half-serpents crossing this edge
(so one does no longer know which head goes with which tail!)

Serpent nests: properties

In any quadrangulation Q, there is only a finite number of serpent nests. This number depends on Q.

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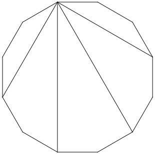

For example,
in this kind of quadrangulation,
serpent nests $\longleftrightarrow$ Dyck paths
(hence counted by Catalan numbers).

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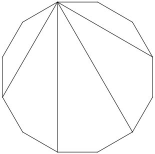

For example,
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conjecture

For any quadrangulation Q, the number of serpent nests in Q is equal to the number of Q-compatible quadrangulations.

This equality can easily be checked for many small examples and for some famillies.

Duality of serpent nests

Serpent nests form a graded set, by the number of serpents, which runs from 0 (empty serpent nest) to $n-1$ (one serpent by edge)

There exists an involution mapping degree k to degree $n-1-k$.

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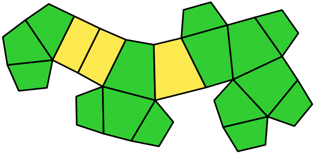
There should also be refined enumerative relations between
F-triangle of Q-compatible quadrangulations
and H-triangle of serpent nests in Q.

Bridges and factorisation

A **bridge** is a square that only has two neighbor squares on opposite sides:



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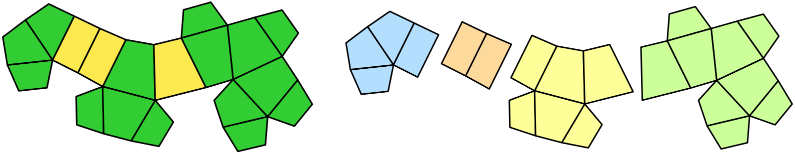
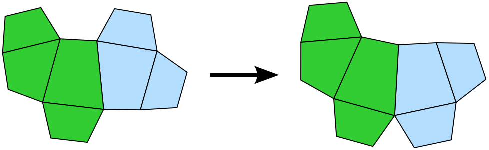


One can show that when there is a bridge,
- the Stoke poset is a product of two Stokes posets,
- the set of serpent nests is also a product.
This last part is because serpents cannot cross the bridges!

Twisting quadrangulations

analog of changing the Coxeter element

Twisting along an edge: operation on quadrangulations



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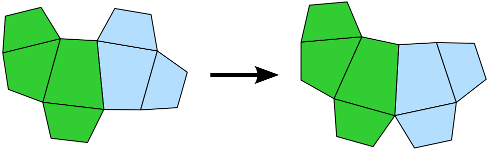

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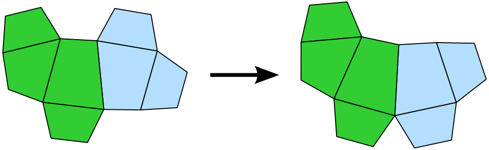


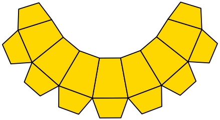
defined by cutting in two parts along one edge,
taking the mirror image of one part and gluing it back.

- this does not change the set of serpent nests (easy bijection)
- It is expected that this does not change the flip graph St_Q.

But the Stokes poset $\overrightarrow{\mathrm{St}}_Q$ does change,
like Cambrian lattices for different Coxeter elements.

A nice familly of examples

There is a nice familly of quadrangulations L_n with $2n$ squares:



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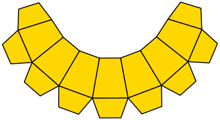

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Their number of serpent nests is given by a Lucas sequence:

$$\ell_0 = 0 \quad \ell_1 = 2 \quad \ell_{n+2} = 6\ell_{n+1} + 3\ell_n,$$

starting $2, 12, 78, 504, 3258, 21060, 136134, 879984, 5688306, \dots$

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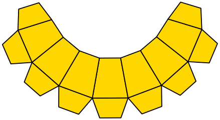

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Maybe the quadrangulations (of even size and with no bridge) with the smallest number of serpent nests.

Open quadrangulations

half-turn symmetry and open serpents

It makes sense, when Q is invariant under half-turn rotation,
with an edge sent to itself by the half-turn,
to speak about invariant Q-compatible quadrangulations.

This should give some type $\mathbb{B}$ objects (flip graphs, posets, polytopes)
including the type $\mathbb{B}$ Cambrian lattices.
One can do the same for serpent nests.

These half-turn invariant serpent nests can be considered as "open"
and one can glue them back by pairs.

Representation theoretic aspects

just a short slide about a long story

Natural context: cluster categories, quiver representations

and also study of derived categories of modules over posets.

- Quadrangulations are **objects** in the derived category of modules over the Tamari lattices.
- The posets $\overrightarrow{\mathrm{St}}_Q$ should describe some **morphisms** between these objects.
- Twisting should not not change the derived category of modules over $\overrightarrow{\mathrm{St}}_Q$.

Moreover, two operads are involved in the story..

Conclusion

To every quadrangulation Q, one associates
- a poset and a polytope, called Stokes poset, Stokes polytope
- a graded set with a duality: serpent nests (but no partial order)

For some specific Q, one recovers type $\mathbb{A}$ cluster combinatorics.
In general, many new generalized "flip graphs".
Some things are lost, for example all the nice product formulas.

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Questions (mysteries)

- Is there something like cluster variables in this setting ?
- What would be the missing noncrossing side of the story ?

The End

감사합니다